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Chiral Extensions of the WZNW Phase Space, Poisson-Lie Symmetries and Groupoids

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Abstract

The chiral WZNW symplectic form Ω^ρ_{chir} is inverted in the general case. Thereby a precise relationship between the arbitrary monodromy dependent 2-form appearing in Ω^ρ_{chir} and the exchange r-matrix that governs the Poisson brackets of the group valued chiral fields is established. The exchange r-matrices are shown to satisfy a new dynamical generalization of the classical modified Yang-Baxter (YB) equation and Poisson-Lie (PL) groupoids are constructed that encode this equation analogously as PL groups encode the classical YB equation. For an arbitrary simple Lie group G, exchange r-matrices are found that are in one-to-one correspondence with the possible PL structures on G and admit them as PL symmetries.

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1 Introduction

This paper contains a systematic analysis of the classical phase space that arises from the chiral separation of the degrees of freedom in the Wess-Zumino-Novikov-Witten (WZNW) model [1]. The WZNW model occupies a central position in conformal field theory [2]. Various structures that emerged from its study play an increasingly important rôle in other areas of theoretical physics and in mathematics as well [3]. Among these structures are the quadratic exchange algebras that encode the Poisson brackets (PBs) of the chiral group valued fields, $g_C(x_C)$ for $C = L, R$, which yield the general solution of the WZNW field equation as $g(x_L, x_R) = g_L(x_L)g_R^{-1}(x_R)$. These exchange algebras were investigated intensively at the beginning of the decade ([4] – [15]) motivated by the idea to understand the quantum group properties of the WZNW model [16] directly by means of canonical quantization [17, 18, 19]. In accordance with the general philosophy of quantum groups [20], the Poisson-Lie (PL) symmetries of the chiral fields should be the most relevant in this respect.

The chiral WZNW Poisson structures found in the literature have the form

$$\Big\{g_C(x) \overset{\otimes}{,} g_C(y)\Big\} = \frac{1}{\kappa_C}\Big(g_C(x) \otimes g_C(y)\Big)\Big(\hat{r} + \frac{1}{2}\hat{I}\,\mathrm{sign}\,(y-x)\Big), \quad 0 < x, y < 2\pi, \tag{1.1}$$

where $\hat{I}$ is given by the quadratic Casimir of the simple Lie algebra, $\mathcal{G}$, of the WZNW group, G, and the interesting object is the 'exchange r-matrix' $\hat{r}$. The choice of the PB is highly non-unique due to the fact that the g_C are determined by the physical field g only up to the gauge freedom $g_C \mapsto g_C p$ for any constant $p \in G$. In general $\hat{r}$ may depend on the monodromy matrix M of the chiral fields, which satisfy $g_C(x + 2\pi) = g_C(x)M$. There are two qualitatively different cases that correspond to building the WZNW field out of chiral fields with diagonal monodromy ('Bloch waves') or out of fields with generic monodromy.

For Bloch waves [8, 9, 10], the Poisson structure is essentially unique and the associated r-matrix is a solution of the so called classical dynamical Yang-Baxter (CDYB) equation, which has recently received a lot of attention (see e.g. the review in [21]).

For chiral fields with generic monodromy, it has been argued in [11, 14] that the possible exchange r-matrices should correspond to certain local differential 2-forms ρ on open domains $\check{G} \subset G$, whose exterior derivative is the 3-form that occurs in the WZNW action. The precise connection between ρ and $\hat{r}$ has not been elaborated, and in most papers dealing with generic monodromy actually only those very special cases were considered for which $\hat{r}$ is a monodromy independent constant. In these cases $\hat{r}$ is necessarily a solution of the classical modified YB equation on the Lie algebra $\mathcal{G}$ with a certain definite normalization (eq. (3.71) with (3.64)). This is a nice situation since if the same r-matrix is used to equip G with a PL structure, then the gauge action of G on the chiral WZNW field defines a PL symmetry. However, this mechanism of PL symmetry is not available in the physically most interesting case of a compact Lie group, because the relevant normalization admits no constant r-matrix for a compact $\mathcal{G}$. Thus, in addition to the problem to understand the case of a general ρ, an interesting question is whether the exchange r-matrix can be chosen for a compact group in such a way to admit a PL symmetry on the chiral WZNW phase space.

In this paper we study the family of chiral exchange algebras (1.1) concentrating on the case of generic monodromy (for a related investigation of Bloch waves, see [22]). Our main results are the following.

- First, we establish the relationship between the 2-form ρ introduced in [11, 14] and the corresponding exchange r-matrix in the general case. The result is given by eq. (3.36) with the notations in (3.35), (3.21), (3.2).

- Second, we point out a dynamical generalization of the modified YB equation, eq. (3.63), whose solutions are the exchange r-matrices for generic monodromy.

- Third, we present explicitly a subfamily of the exchange r-matrices whose members are in one-to-one correspondence with the possible PL structures on G and admit them as PL symmetries. These exchange r-matrices, given by (5.18) with (5.22), contain the constant r-matrices studied earlier, and in another remarkable special case of them the gauge action of the group G becomes a classical symmetry, i.e., PL symmetry with the zero Poisson structure on G. They work for any (compact or not) simple Lie group.

- Fourth, we construct a family of PL groupoids that encode the dynamical YB equation (3.63) analogously as PL groups encode the classical YB equation. This result generalizes a construction of [23] from diagonal to generic monodromy.

The above mentioned results have been announced in [24] without proofs. In addition to their detailed account, several other technical results can be found in this paper. The systematic exposition of the subject and the numerous examples that we present may be useful as a starting point for future studies.

The organization of the rest of the paper is as follows. In the next section a necessary review of the chiral separation of the WZNW phase space is presented. Section 3 contains a detailed account of the inversion of the possible symplectic structures on the chiral WZNW phase space, leading to the exchange algebra (1.1). Here many interesting additional issues are considered as well. In section 4 an alternative, shorter but less rigorous, derivation of the general exchange algebra is given, and a quick derivation of the exchange algebra of Bloch waves is also included. Section 5 is devoted to a general outline of the PL symmetries of the exchange algebra, and in particular to the exchange r-matrices for which the standard gauge action of G on the chiral WZNW field yields such a symmetry. Section 6 deals with the interpretation of the chiral WZNW Poisson structures in terms of PL groupoids. The paper ends with a discussion, and there are also two appendices containing some examples and the details of a proof.

2 The WZNW solution space and its chiral extension

In this section we review the structure of the WZNW Hamiltonian system concentrating on the possible symplectic forms on the chiral extension of its solution space, which are examined throughout the paper. The presentation closely follows the line of thought found in [14].

We consider a simple, real or complex, Lie algebra, $\mathcal{G}$, with a corresponding connected Lie group, G, and identify the phase space of the WZNW model associated with the group G as

$$\mathcal{M} = T^*\widetilde{G} = \{\, (g, J_L) \,|\, g \in \widetilde{G},\ \ J_L \in \widetilde{\mathcal{G}}\,\}, \tag{2.1}$$

where $\widetilde{G} = C^\infty(S^1, G)$ is the loop group and $\widetilde{\mathcal{G}} = C^\infty(S^1, \mathcal{G})$ is its Lie algebra. The isomorphism of the cotangent bundle $T^*\widetilde{G}$ with $\widetilde{G}\times\widetilde{\mathcal{G}}$ is established by means of right-translations on $\widetilde{G}$. The elements $g \in \widetilde{G}$ (resp. $J_L \in \widetilde{\mathcal{G}}$) are modeled as 2π-periodic G-valued (resp. $\mathcal{G}$-valued) functions on the real line $\mathbf{R}$. The phase space is equipped with the symplectic form

$$\Omega^\kappa = d\int_0^{2\pi} d\sigma\, \mathrm{Tr}\left(J_L dg g^{-1}\right) + \frac{\kappa}{2}\int_0^{2\pi} d\sigma\, \mathrm{Tr}\left(dg g^{-1}\right)\wedge\left(dg g^{-1}\right)' \tag{2.2}$$

with some constant κ. Here prime denotes derivative with respect to the space variable, $\sigma \in \mathbf{R}$, and for any $A, B \in \mathcal{G}$ $\mathrm{Tr}\,(AB)$ denotes a fixed multiple of the Cartan-Killing scalar product. If g and the 'left-current' J_L serve as coordinates on $\mathcal{M}$, then the 'right-current' is given by

$$J_R = -g^{-1}J_L g + \kappa g^{-1}g', \tag{2.3}$$

and in the alternative variables (g, J_R) the symplectic form reads

$$\Omega^\kappa = -d\int_0^{2\pi} d\sigma\, \mathrm{Tr}\left(J_R g^{-1}dg\right) - \frac{\kappa}{2}\int_0^{2\pi} d\sigma\, \mathrm{Tr}\left(g^{-1}dg\right)\wedge\left(g^{-1}dg\right)'. \tag{2.4}$$

Although the expression of Ω^κ appears rather formal at first sight, it can be used to unambiguously associate hamiltonian vector fields and PBs with a set of admissible functions, which include, for example, the Fourier components of g, J_L and J_R. We do not elaborate the precise meaning of the symplectic form here, since this is a standard matter in the context of the full WZNW model, but will face the analogous issue in the chiral context later, where it is much less understood. The only point that we wish to note is that in the case of a complex Lie algebra the admissible functions depend holomorphically on the matrix elements of g, J_L, J_R in the finite dimensional irreducible representations of G, and $\widetilde{G}\times\widetilde{\mathcal{G}}$ is then a model of the holomorphic cotangent bundle.

The phase space $\mathcal{M}$ represents the initial data for the WZNW system, whose dynamics is generated by the Hamiltonian

$$H_{\mathrm{WZNW}} = \frac{1}{2\kappa}\int_0^{2\pi} d\sigma\, \mathrm{Tr}\left(J_L^2 + J_R^2\right). \tag{2.5}$$

Denoting time by τ and introducing lightcone coordinates as

$$x_L := \sigma+\tau, \quad x_R := \sigma-\tau, \quad \partial_L = \frac{\partial}{\partial x_L} = \frac{1}{2}(\partial_\sigma + \partial_\tau), \quad \partial_R = \frac{\partial}{\partial x_R} = \frac{1}{2}(\partial_\sigma - \partial_\tau), \tag{2.6}$$

Hamilton's equation can be written in the alternative forms [1]

$$\kappa\partial_L g = J_L g, \quad \partial_R J_L = 0 \qquad \Leftrightarrow \qquad \kappa\partial_R g = gJ_R, \quad \partial_L J_R = 0. \tag{2.7}$$

Let $\mathcal{M}^{sol}$ be the space of solutions of the WZNW system. $\mathcal{M}^{sol}$ consists of the smooth G-valued functions $g(\sigma,\tau)$ which are 2π-periodic in σ and satisfy $\partial_R(\partial_L g\, g^{-1}) = 0$. The general solution of this evolution equation can be written as

$$g(\sigma,\tau) = g_L(x_L)g_R^{-1}(x_R), \tag{2.8}$$

where (g_L, g_R) is any pair of G-valued, smooth, quasiperiodic function on $\mathbf{R}$ with *equal monodromies*, i.e., for $C = L, R$ one has $g_C(x_C+2\pi) = g_C(x_C)M$ with some C-independent $M \in G$. To elaborate this representation of the solutions in more detail, we define the space $\widehat{\mathcal{M}}$:

$$\widehat{\mathcal{M}} := \{(g_L, g_R) | g_{L,R} \in C^\infty(\mathbf{R}, G), \quad g_{L,R}(x+2\pi) = g_{L,R}(x)M \quad M \in G\}. \tag{2.9}$$

There is a free right-action of G on $\widehat{\mathcal{M}}$ given by

$$G \ni p : (g_L, g_R) \mapsto (g_L p, g_R p). \tag{2.10}$$

Notice that $\widehat{\mathcal{M}}$ is a *principal fibre bundle* over $\mathcal{M}^{sol}$ with respect to the above action of G. The projection of this bundle, $\vartheta : \widehat{\mathcal{M}} \to \mathcal{M}^{sol}$, is given by

$$\vartheta : (g_L, g_R) \mapsto g = g_L g_R^{-1} \quad \text{i.e.} \quad g(\sigma, \tau) = g_L(x_L) g_R^{-1}(x_R). \tag{2.11}$$

We can identify $\mathcal{M}$ with $\mathcal{M}^{sol}$ by associating the elements of the solution space with their initial data at $\tau = 0$. Formally, this is described by the map

$$\iota : \mathcal{M}^{sol} \to \mathcal{M}, \qquad \iota : \mathcal{M}^{sol} \ni g(\sigma, \tau) \mapsto \Big(g(\sigma, 0), J_L(\sigma) = (\kappa \partial_L g\, g^{-1})(\sigma, 0) \Big) \in \mathcal{M}. \tag{2.12}$$

Obviously, $\iota^*(\Omega^\kappa)$ is then the natural symplectic form on the solution space. Explicitly,

$$(\iota^* \Omega^\kappa)(g) = -\kappa \left(d \int_0^{2\pi} d\sigma \, \mathrm{Tr} \left(g^{-1} \partial_R g \, g^{-1} dg \right) + \frac{1}{2} \int_0^{2\pi} d\sigma \, \mathrm{Tr} \left(g^{-1} dg \right) \wedge \partial_\sigma \left(g^{-1} dg \right) \right) |_{\tau=0} \tag{2.13}$$

Regarding now $\mathcal{M}^{sol}$ as the base of the bundle $\vartheta : \widehat{\mathcal{M}} \to \mathcal{M}^{sol}$, we obtain a closed 2-form, $\widehat{\Omega}^\kappa$, on $\widehat{\mathcal{M}}$,

$$\widehat{\Omega}^\kappa := \vartheta^*(\iota^* \Omega^\kappa) = (\iota \circ \vartheta)^* \Omega^\kappa. \tag{2.14}$$

By substituting the explicit formula (2.11) of ϑ, one finds that

$$\widehat{\Omega}^\kappa(g_L, g_R) = \kappa_L \Omega_{chir}(g_L) + \kappa_R \Omega_{chir}(g_R), \tag{2.15}$$

where

$$\kappa_L := \kappa, \qquad \kappa_R := -\kappa, \tag{2.16}$$

and Ω_{chir} is the so called chiral WZNW 2-form:

$$\begin{aligned} &\Omega_{chir}(g_C) = -\frac{1}{2} \int_0^{2\pi} dx_C \, \mathrm{Tr} \left(g_C^{-1} dg_C \right) \wedge \left(g_C^{-1} dg_C \right)' - \frac{1}{2} \mathrm{Tr} \left((g_C^{-1} dg_C)(0) \wedge dM_C \, M_C^{-1} \right), \\ &M_C = g_C^{-1}(x) g_C(x + 2\pi). \end{aligned} \tag{2.17}$$

This crucial formula of $\widehat{\Omega}^\kappa$ was first obtained by Gawedzki [11].

It is clear from its definition that $d\widehat{\Omega}^\kappa = 0$, but $\widehat{\Omega}^\kappa$ is not a symplectic form on $\widehat{\mathcal{M}}$, since it is degenerate. Of course, its restriction to any (local) section of the bundle $\vartheta : \widehat{\mathcal{M}} \to \mathcal{M}^{sol}$ is a symplectic form, since such sections yield (local) models of $\mathcal{M}^{sol}$. On the other hand, one can check that Ω_{chir} has a nonvanishing exterior derivative [11]:

$$d\Omega_{chir}(g_C) = -\frac{1}{6} \mathrm{Tr} \left(M_C^{-1} dM_C \wedge M_C^{-1} dM_C \wedge M_C^{-1} dM_C \right). \tag{2.18}$$

Although this cancels from $d\widehat{\Omega}^\kappa$, since $M_L = M_R$ for the elements of $\widehat{\mathcal{M}}$, it makes the chiral separation of the WZNW degrees of freedom a very nontrivial and interesting problem.

The problem of the chiral separation can be described as follows [14]. First, recall that the chiral currents J_C ($C = L, R$) generate two commuting copies of the nontwisted affine Kac-Moody (KM) algebra of $\mathcal{G}$ and the WZNW field (2.8) is a KM primary field under the Poisson bracket defined by the symplectic form on $\mathcal{M}^{sol}$. In fact, by defining Fourier components as

$$J_C^{\alpha,n} := \int_0^{2\pi} dx_C \, e^{-inx_C} \mathrm{Tr}\,(T^\alpha J_C)(x_C) \tag{2.19}$$

using a basis[1] T^α of $\mathcal{G}$, it can be derived from (2.13) that the currents satisfy

$$\{J_C^{\alpha,m}, J_C^{\beta,n}\} = f^{\alpha\beta}_\gamma J_C^{\gamma,m+n} + 2i\pi\kappa_C m\delta_{m,-n} I^{\alpha\beta}, \qquad \{J_L^{\alpha,m}, J_R^{\beta,n}\} = 0, \tag{2.20}$$

and

$$\{g(x_L, x_R), J_L^{\alpha,n}\} = e^{-inx_L} T^\alpha g(x_L, x_R), \qquad \{g(x_L, x_R), J_R^{\alpha,n}\} = -e^{-inx_R} g(x_L, x_R) T^\alpha. \tag{2.21}$$

Second, the currents *almost* completely determine the chiral WZNW fields g_C, and thus also $g = g_L g_R^{-1}$, by means of the differential equations

$$\kappa_C \partial_C g_C = J_C g_C \quad \text{for} \quad C = L, R. \tag{2.22}$$

Thus it appears an interesting possibility to construct the WZNW model as a reduction of a simpler model, in which the left and right-moving degrees of freedom would be separated in terms of *completely independent* chiral fields g_L and g_R regarded as fundamental variables. It is clear that the solution space of such a chirally extended model must be a direct product of two identical but independent spaces, i.e., it must have the form

$$\widehat{\mathcal{M}}^{ext} := \mathcal{M}_L \times \mathcal{M}_R \tag{2.23}$$

with

$$\mathcal{M}_C := \{g_C \,|\, g_C \in C^\infty(\mathbf{R}, G), \quad g_C(x + 2\pi) = g_C(x) M_C \quad M_C \in G\}. \tag{2.24}$$

The space $\widehat{\mathcal{M}}^{ext}$ must be endowed with such a symplectic structure, $\hat{\Omega}^\kappa_{ext}$, that reduces to $\hat{\Omega}^\kappa$ on the submanifold $\widehat{\mathcal{M}} \subset \widehat{\mathcal{M}}^{ext}$ defined by the periodicity constraint

$$M_L = M_R. \tag{2.25}$$

It is easy to see that these requirements force $\hat{\Omega}^\kappa_{ext}$ to have the following form:

$$\hat{\Omega}^\kappa_{ext}(g_L, g_R) = \kappa_L \Omega^\rho_{chir}(g_L) + \kappa_R \Omega^\rho_{chir}(g_R) \tag{2.26}$$

where

$$\Omega^\rho_{chir}(g_C) = \Omega_{chir}(g_C) + \rho(M_C) \tag{2.27}$$

with some 2-form ρ depending *only* on the monodromy of g_C. Since in the extended model the factors $(\mathcal{M}_C, \kappa_C \Omega^\rho_{chir})$ must be *symplectic* manifolds *separately*, we have to satisfy the condition

$$d\Omega^\rho_{chir} = -\frac{1}{6}\mathrm{Tr}\,\left(M_C^{-1} dM_C \wedge M_C^{-1} dM_C \wedge M_C^{-1} dM_C\right) + d\rho(M_C) = 0. \tag{2.28}$$

[1] We have $I^{\alpha\beta} := \mathrm{Tr}\,(T^\alpha T^\beta)$ and $[T^\alpha, T^\beta] = f^{\alpha\beta}_\gamma T^\gamma$ with summation over coinciding indices.

The problem now arises from the well-known fact that no globally defined smooth 2-form exists on G that would satisfy this condition for all $M_C \in G$.

There are two rather different wayouts from the above difficulty [14]. The first is to restrict the possible domain of the monodromy matrix M_C to some open submanifold in G on which an appropriate 2-form ρ may be found. We refer to a choice of such a domain and a ρ as *a chiral extension of the WZNW system*, and will explore the structure of the associated PB in the subsequent sections.

The second possibility is to restrict the domain of the allowed monodromy matrices much more drastically from the beginning, in such a way that after the restriction $d\Omega_{chir}$ vanishes, whereby the difficulty disappears. For example, one may achieve this by restricting the monodromy matrices to vary in a fixed maximal torus of G, which amounts to constructing (a subset of) the solutions of the WZNW field equation in terms of chiral 'Bloch waves'. This second possibility is especially natural in the case of compact or complex Lie groups, for which there is only one maximal torus up to conjugation. The restriction to Bloch waves is equivalent to a partial (and local) gauge fixing of the bundle $\vartheta : \widehat{\mathcal{M}} \to \mathcal{M}^{sol}$. The resulting symplectic form is studied in detail in [22].

3 The chiral WZWN phase space

We here investigate the structure of the chiral WZNW phase space $\mathcal{M}_C$ introduced in sec. 2. The analysis is the same for both chiralities, $C = L, R$, and we simplify our notation by putting $\mathcal{M}_{chir}$ for $\mathcal{M}_C$ and g, M, J, κ for g_C, M_C, J_C, κ_C, respectively. We assume that the monodromy matrix M is restricted to some open submanifold $\check{G} \subset G$ on which a smooth 2-form ρ is chosen in such a way that (2.28) holds. The domain in $\mathcal{M}_{chir}$ that corresponds to $M \in \check{G}$ is denoted by $\check{\mathcal{M}}_{chir}$. It turns out that $\kappa\Omega^{\rho}_{chir}$, defined by (2.27) with (2.17), is nondegenerate if $\check{G}$ is appropriately chosen (so that eq. (3.34) has a smooth, unique solution), and we shall describe the general features of the PBs on $\check{\mathcal{M}}_{chir}$ associated with this symplectic form. We will then consider examples, in particular the choices of ρ introduced in [14] that lead to Poisson-Lie symmetry on the full $\mathcal{M}_{chir}$.

3.1 Lie algebraic and differential geometric conventions

Before we can turn to the task of inverting $\kappa\Omega^{\rho}_{chir}$, we need to set up some conventions.

An element $A \in \mathcal{G}$ has the components $A_\alpha = \mathrm{Tr}\,(AT_\alpha)$ and $A^\alpha = \mathrm{Tr}\,(AT^\alpha)$ with respect to dual bases T_α and T^β of $\mathcal{G}$:

$$\mathrm{Tr}\,(T_\alpha T^\beta) = \delta^\beta_\alpha, \qquad I^{\alpha\beta} = \mathrm{Tr}\,(T^\alpha T^\beta), \qquad I_{\alpha\beta} = \mathrm{Tr}\,(T_\alpha T_\beta). \tag{3.1}$$

We will use $I^{\alpha\beta}$ and $I_{\alpha\beta}$ to raise and lower Lie algebra indices. Given a matrix $Q_{\alpha\beta}$, we can define an operator $Q \in \mathrm{End}(\mathcal{G})$ and an element $\hat{Q} \in \mathcal{G} \otimes \mathcal{G}$ by

$$Q(A) = T_\alpha Q^{\alpha\beta} A_\beta, \qquad \hat{Q} = Q^{\alpha\beta} T_\alpha \otimes T_\beta. \tag{3.2}$$

The matrix $I^{\alpha\beta}$ defines the identity operator I, and $\hat{I} = T_\alpha \otimes T^\alpha$ is the 'tensor Casimir'. For any $M \in G$, the matrix of the linear operator $\mathrm{Ad}\, M$ on $\mathcal{G}$, which we write as $(\mathrm{Ad}\, M)(A) = MAM^{-1}$, will be denoted as

$$W_{\alpha\beta}(M) = \mathrm{Tr}\,(T_\alpha M T_\beta M^{-1}). \tag{3.3}$$

The property $(\mathrm{Ad}\,M)(\mathrm{Ad}\,\tilde{M}) = (\mathrm{Ad}\,M\tilde{M})$ yields the matrix multiplication rule

$$W_{\alpha\gamma}(M) I^{\gamma\theta} W_{\theta\beta}(\tilde{M}) = W_{\alpha\beta}(M\tilde{M}), \tag{3.4}$$

and we also have

$$W_{\beta\alpha}(M) = W_{\alpha\beta}(M^{-1}) = W^{-1}_{\alpha\beta}(M), \qquad W_{\alpha\beta}(M) I^{\beta\gamma} W^{-1}_{\gamma\theta}(M) = I_{\alpha\theta}. \tag{3.5}$$

Acting on a smooth function ψ on G, we introduce the differential operators

$$(\mathcal{L}_\alpha \psi)(M) := \frac{d}{dt}\psi(e^{tT_\alpha}M)\Big|_{t=0}, \qquad (\mathcal{R}_\alpha \psi)(M) := \frac{d}{dt}\psi(Me^{tT_\alpha})\Big|_{t=0}, \tag{3.6}$$

and their linear combinations

$$\mathcal{D}^{\pm}_\alpha := \mathcal{R}_\alpha \pm \mathcal{L}_\alpha. \tag{3.7}$$

For some purposes we will use a representation $\Lambda : G \to GL(V)$ of G on a finite dimensional vector space V. The corresponding representation of $\mathcal{G}$ is denoted by the same letter, and we put

$$M^\Lambda := \Lambda(M) \quad \text{for} \quad M \in G, \qquad T^\Lambda := \Lambda(T) \quad \text{for} \quad T \in \mathcal{G}. \tag{3.8}$$

Such representations will be used (e.g. the irreducible ones) that

$$\mathrm{Tr}\,(T_\alpha T_\beta) = c_\Lambda \mathrm{tr}\,(T^\Lambda_\alpha T^\Lambda_\beta) \tag{3.9}$$

holds with a constant c_Λ, and we will then write

$$\mathrm{Tr}\,(AB) := c_\Lambda \mathrm{tr}\,(AB), \qquad \forall A, B \in \mathrm{End}(V). \tag{3.10}$$

On the left hand side of eq. (3.9) Tr is a fixed (representation independent) multiple of the Cartan-Killing form of $\mathcal{G}$, while tr is matrix trace over the representation space V.

Remember that the phase space $\mathcal{M}_{chir}$ is parametrized by the G-valued field $g(x)$, which is assumed to be smooth in x and is subject to the monodromy condition

$$g(x + 2\pi) = g(x)M \qquad M \in G. \tag{3.11}$$

The corresponding chiral current, $J(x) = \kappa g'(x) g^{-1}(x) \in \mathcal{G}$, is thus a smooth, 2π-periodic function of x. To define tangent vectors at $g \in \mathcal{M}_{chir}$, we first have to consider smooth curves on the phase space. Such a curve is given by a function $\gamma(x,t) \in G$, which is smooth in x, t and satisfies the deformed monodromy condition

$$\gamma(x + 2\pi, t) = \gamma(x,t)M(t) \qquad M(t) \in G. \tag{3.12}$$

To make sure that the curve passes through $g \in \mathcal{M}_{chir}$ at $t = 0$, we require $\gamma(x,0) = g(x)$. A vector $X[g]$ at $g \in \mathcal{M}_{chir}$ is obtained as the velocity to the curve at $t = 0$, encoded by the $\mathcal{G}$-valued, smooth function

$$\xi(x) := \frac{d}{dt} g^{-1}(x)\gamma(x,t)\Big|_{t=0}. \tag{3.13}$$

It is useful to note that, due to the analogous property of g, the function ξ on $\mathbf{R}$ may be reconstructed from its restriction to $[0, 2\pi]$. The monodromy properties of $\xi(x)$ can be derived by taking the derivative of (3.12),

$$\xi'(x + 2\pi) = M^{-1}\xi'(x)M. \tag{3.14}$$

This can be solved in terms of a smooth, 2π-periodic $\mathcal{G}$-valued function, $X_J(x)$, and a constant Lie algebra element, ξ_0, as follows:

$$\xi(x) = \xi_0 + \int_0^x dy\, g^{-1}(y)X_J(y)g(y). \tag{3.15}$$

A vector field X on $\mathcal{M}_{chir}$ is an assignment, $g \mapsto X[g]$, of a vector to every point $g \in \mathcal{M}_{chir}$. It acts on a differentiable function, $g \mapsto F[g]$, on $\mathcal{M}_{chir}$ by the definition

$$X(F)[g] = \frac{d}{dt}F[g_t]\Big|_{t=0} \qquad g_t(x) = \gamma(x,t). \tag{3.16}$$

Since a vector $X[g]$ can be parametrized by $\xi(x)$, which, in turn, can be parametrized by the pair $(\xi_0, X_J(x))$, we can specify a vector field by the assignments $g \mapsto \xi_0[g] \in \mathcal{G}$ and $g \mapsto X_J[g] \in \widetilde{\mathcal{G}}$. Of course, the evaluation functions $F^x[g] := g(x)$ and $\mathcal{F}^x[g] := J(x)$ are differentiable with respect to any vector field, and their derivatives are given by

$$X(g(x)) = g(x)\xi(x) \qquad \text{and} \qquad X(J(x)) = \kappa X_J(x), \tag{3.17}$$

which clarifies the meaning of X_J as well. It is also obvious from (3.11) that the monodromy matrix yields a G-valued differentiable function on $\mathcal{M}_{chir}$, $g \mapsto M = g^{-1}(x)g(x+2\pi)$, whose derivative is characterized by the $\mathcal{G}$-valued function

$$X(M)M^{-1} = M\xi(x+2\pi)M^{-1} - \xi(x). \tag{3.18}$$

Having defined vector fields, one can now introduce differential forms as usual. We only remark that by (3.17) evaluation 1-forms like $dg(x)$, $dJ(x)$ or $(g^{-1}dg)'(x)$ are perfectly well-defined:

$$\begin{aligned} &dg(x)(X) = X(g(x)) = g(x)\xi(x), \quad dJ(x)(X) = X(J(x)) = \kappa X_J(x), \\ &(g^{-1}(x)dg(x))'(X) = \xi'(x). \end{aligned} \tag{3.19}$$

3.2 Admissible Hamiltonians and hamiltonian vector fields

Now we turn to the following problem. For a fixed (scalar) function F on the phase space $\check{\mathcal{M}}_{chir}$, we are looking for a corresponding vector field, Y^F, satisfying

$$X(F) = \kappa\Omega^{\rho}_{chir}(X, Y^F) \tag{3.20}$$

for all vector fields X. Notice that Y^F does not necessarily exist for a given F. We say that F is an element of the set of *admissible Hamiltonians*, denoted as $\mathtt{H}$, if the corresponding hamiltonian vector field, Y^F, exists. Our purpose is to characterize $\mathtt{H}$ and to describe the mapping $\mathtt{H} \ni F \mapsto Y^F$.

We first compute $\kappa\Omega^\rho_{chir}(X,Y)$ for two vector fields X and Y. Let X be parametrized by $\xi(x)$ and further by the pair $(\xi_0, X_J(x))$. The analogous parametrization for Y is given by $\eta(x)$ and the pair $(\eta_0, Y_J(x))$. Recall that $\kappa\Omega^\rho_{chir}$ is defined by eqs. (2.17), (2.27), and parametrize ρ now as

$$\rho(M) = \frac{1}{2} q^{\alpha\beta}(M)\mathrm{Tr}(T_\alpha M^{-1}dM) \wedge \mathrm{Tr}(T_\beta M^{-1}dM). \tag{3.21}$$

The $q^{\alpha\beta}$, $q^{\alpha\beta} = -q^{\beta\alpha}$, are smooth functions on the domain $\check{G} \subset G$, such that $d\Omega^\rho_{chir} = 0$ on $\check{\mathcal{M}}_{chir}$. A simple calculation, using partial integrations and (3.18), gives that

$$\kappa\Omega^\rho_{chir}(X,Y) = \kappa\Omega_{chir}(X,Y) + \kappa\rho(X,Y), \tag{3.22}$$

where

$$\begin{aligned} \Omega_{chir}(X,Y) &= \int_0^{2\pi} dx \mathrm{Tr}\left(X_J(x)g(x)\Big(\eta(x) - \frac{1}{2}M^{-1}Y(M)\Big)g^{-1}(x)\right) \\ &- \frac{1}{2}\mathrm{Tr}\left(\xi_0\Big(M^{-1}Y(M) + Y(M)M^{-1}\Big)\right) \end{aligned} \tag{3.23}$$

and

$$\rho(X,Y) = \int_0^{2\pi} dx \mathrm{Tr}\left(X_J(x)g(x)B^Y(M)g^{-1}(x)\right) + \mathrm{Tr}\left(\xi_0\left(B^Y(M) - MB^Y(M)M^{-1}\right)\right) \tag{3.24}$$

with

$$B^Y(M) := q^{\alpha\beta}(M)T_\alpha \mathrm{Tr}\left(T_\beta M^{-1}Y(M)\right). \tag{3.25}$$

Of course, all the expressions that appear in the above formula are functions of $g \in \check{\mathcal{M}}_{chir}$.

Let us now suppose that $F \in \mathtt{H}$ and apply the above formula to $Y := Y^F$. Then the form of the right hand side of (3.20) implies that there must exist a *smooth* $\mathcal{G}$-valued function on $\mathbf{R}$, $A^F(x)$, and a constant Lie algebra element, a^F, such that for any vector field X

$$X(F) = \kappa \int_0^{2\pi} dx \mathrm{Tr}\left(X_J(x)A^F(x)\right) + \kappa\mathrm{Tr}\,(\xi_0 a^F). \tag{3.26}$$

This means that $F \in \mathtt{H}$ must have an exterior derivative parametrized by the assignments $g \mapsto A^F(x)[g]$ and $g \mapsto a^F[g]$. On the other hand, if F is such that (3.26) holds, then we may try to solve (3.20) for the hamiltonian vector field. Using the parametrization of Y by $\eta(x)$, this leads to the following two equations:

$$\eta(x) - \frac{1}{2}M^{-1}Y(M) + B^Y(M) = g^{-1}(x)A^F(x)g(x), \tag{3.27}$$

and

$$M^{-1}Y(M) + Y(M)M^{-1} - 2B^Y(M) + 2MB^Y(M)M^{-1} = -2a^F. \tag{3.28}$$

Now it is clear that $\eta'(x)$ can be directly read off from (3.27). From the identity $\eta'(x) = g^{-1}(x)Y_J(x)g(x)$, we then obtain that

$$Y_J(x) = A^{F\prime}(x) + \frac{1}{\kappa}\Big[A^F(x), J(x)\Big]. \tag{3.29}$$

This not only gives us the explicit formula of Y_J, but also means that any $F \in \mathtt{H}$ must be such that the right hand side of (3.29) defines a 2π-*periodic* smooth function of x. Incidentally, this is equivalent to the monodromy condition (3.14) applied to the hamiltonian vector field Y. To proceed further, we use

$$Y(M)M^{-1} = M\eta(2\pi)M^{-1} - \eta(0). \tag{3.30}$$

This implies that eqs. (3.27) and (3.28) are not linearly independent, and they can be simultaneously solved for $\eta(x)$ only if one has

$$a^F = g^{-1}(0)\Big[A^F(0) - A^F(2\pi)\Big]g(0). \tag{3.31}$$

We now also see that the pair (A^F, a^F) is uniquely determined for any $F \in \mathtt{H}$. Indeed, the restriction of A^F to $[0, 2\pi]$ is completely fixed by (3.26), and is uniquely extended to a function on $\mathbf{R}$ on account of the periodicity of the expression in (3.29).

To summarize, we have shown that every element $F \in \mathtt{H}$ must satisfy the three conditions[2] expressed by (3.26), the periodicity of $Y_J(x)$ in (3.29), and (3.31). Conversely, it turns out that these conditions characterize $\mathtt{H}$. In fact, if these conditions are satisfied then the solution of (3.27), (3.28) for η is given by

$$\eta(x) = g^{-1}(x)A^F(x)g(x) - \frac{1}{2}a^F + r(M)(a^F)\,, \tag{3.32}$$

where

$$r(M)(a^F) = T_\alpha r^{\alpha\beta}(M)a^F_\beta \tag{3.33}$$

and the matrix $r^{\alpha\beta}$ *is defined as the solution of the linear equation*

$$r^{\alpha\beta} + \left(W_\gamma{}^\alpha - 2q_\gamma{}^\alpha - 2q^{\alpha\theta}W_{\gamma\theta}\right)r^{\gamma\beta} = \frac{1}{2}I^{\alpha\beta} - \frac{1}{2}W^{\beta\alpha} + q^{\alpha\beta} - q_\gamma{}^\alpha W^{\beta\gamma}. \tag{3.34}$$

This formula of $\eta(x) = g^{-1}(x)Y^F(g(x))$ is one of the main results in this paper.

Some remarks are here in order. First, in (3.34) we suppressed the M-dependence of the various matrices like $q_\gamma{}^\alpha(M) = q^{\beta\alpha}(M)I_{\beta\gamma}$ and $W_\gamma{}^\alpha = \mathrm{Tr}\,(T_\gamma M T^\alpha M^{-1})$. Second, in terms of the notations given at the beginning of the section, $r(M)$ is the linear operator on $\mathcal{G}$ associated with the matrix $r^{\alpha\beta}(M)$. By introducing now the operators $r_\pm(M)$ and $q_\pm(M)$ that correspond to the matrices

$$r^{\alpha\beta}_\pm(M) = r^{\alpha\beta}(M) \pm \frac{1}{2}I^{\alpha\beta} \quad \text{and} \quad q^{\alpha\beta}_\pm(M) = q^{\alpha\beta}(M) \pm \frac{1}{2}I^{\alpha\beta}, \tag{3.35}$$

eq. (3.34) can be rewritten, in fact, in the following equivalent form:

$$q_+(M) \circ r_-(M) = q_-(M) \circ \mathrm{Ad}\,(M^{-1}) \circ r_+(M). \tag{3.36}$$

The solution can be formally written as

$$r(M) = \frac{1}{2}\left(q_+(M) - q_-(M) \circ \mathrm{Ad}\,(M^{-1})\right)^{-1} \circ \left(q_+(M) + q_-(M) \circ \mathrm{Ad}\,(M^{-1})\right). \tag{3.37}$$

[2]These conditions do not depend on the 2-form ρ, a reason for this is described at the end of sec. 3.5.

This shows that one must *define* the domain $\check{G}$ in such a way that the inverse operator above exists, which is always possible since it becomes the identity operator at $M = e \in G$. Then it is easy to see that (3.37) yields a smooth, *antisymmetric* matrix function $r^{\alpha\beta}(M)$ on $\check{G}$. By choosing $\check{G} \subset G$ appropriately, hence we may indeed associate with the smooth 2-form ρ on $\check{G}$ a unique, smooth map $\check{G} \ni M \mapsto r(M) \in \mathrm{End}(\mathcal{G})$. Third, we will see in the next section that the object

$$\hat{r}(M) = r^{\alpha\beta}(M) T_\alpha \otimes T_\beta \in \mathcal{G} \wedge \mathcal{G} \tag{3.38}$$

appears in the classical exchange algebra that encodes the Poisson brackets corresponding to the symplectic form $\kappa\Omega^\rho_{chir}$ and it satisfies a dynamical generalization of the modified classical Yang-Baxter equation. Incidentally, $\kappa\Omega^\rho_{chir}$ is *symplectic* (i.e. nondegenerate) in the sense that it permits to unambiguously determine the map $\mathtt{H} \ni F \mapsto Y^F$, as we just saw, and $\mathtt{H}$ will turn out to contain a 'complete set of functions' on $\check{\mathcal{M}}_{chir}$.

Finally, as for the derivation of eq. (3.34), note that one may arrive at the special form of the integration constant in formula (3.32) of $\eta(x)$ by the expectation of a classical exchange algebra type PB for the field $g(x)$, or simply by inspecting the equations that result if one writes $\eta(x) = g^{-1}(x) A^F(x) g(x) + \text{constant}$. After introducing (3.32) as an ansatz, it is not difficult to verify that (3.27) and (3.28) reduce to (3.34).

3.3 Elements of H and their Poisson brackets

Below we describe a large set of functions that are admissible Hamiltonians and apply the result in (3.32) to find their hamiltonian vector fields. We shall also discuss the interpretation of these hamiltonian vector fields in terms of PBs, in particular we shall see that the field $g(x)$ is subject to a quadratic exchange algebra.

Let us first study functions that depend on g only through the current $J = \kappa g' g^{-1}$. Of course, the evaluation functions $\mathcal{F}^y_\alpha[g] = J_\alpha(y)$ do not belong to $\mathtt{H}$, since $A^{\mathcal{F}^y_\alpha}(x)$ in (3.26) would not be a *smooth* function of x. Therefore we consider the 'smeared out' version

$$\mathcal{F}_\mu := \int_0^{2\pi} dx \mathrm{Tr}\left(\mu(x) J(x)\right), \tag{3.39}$$

where $\mu(x)$ is a $\mathcal{G}$-valued, smooth, 2π-periodic test function. In this case we find that

$$A^{\mathcal{F}_\mu}(x) = \mu(x) \qquad \text{and} \qquad a^{\mathcal{F}_\mu} = 0. \tag{3.40}$$

The conditions expressed by (3.26), (3.29) and (3.31) are trivially satisfied and thus $\mathcal{F}_\mu \in \mathtt{H}$. The parameter $\eta(x)$ of the hamiltonian vector field $Y^{\mathcal{F}_\mu}$ is $\eta(x) = g^{-1}(x)\mu(x)g(x)$, whence

$$Y^{\mathcal{F}_\mu}(g(x)) = \mu(x) g(x), \quad Y^{\mathcal{F}_\mu}(J(x)) = [\mu(x), J(x)] + \kappa\mu'(x), \qquad Y^{\mathcal{F}_\mu}(M) = 0. \tag{3.41}$$

This shows in particular that $\mathcal{F}_\mu$ generates an infinitesimal action of the loop group on the phase space with respect to which $g(x)$ is an affine KM primary field, and the KM current $J(x)$ transforms according to the coadjoint action of the (centrally extended) loop group, as expected. Naturally, the *local* functionals of J defined as the integral over $[0, 2\pi]$ of any differential polynomial in the components of J, with periodic, smooth test function coefficients, also belong to $\mathtt{H}$; the corresponding hamiltonian vector fields are easy to determine.

Now we study some *nonlocal* functionals of the current. Let $\mathcal{E} \in G$ denote the path ordered exponential integral of $J(x)$ over $[0, 2\pi]$. More precisely, we put $\mathcal{E} := E(2\pi)$, where $E(x) \in G$ is defined as the solution of

$$\kappa E'(x) = J(x)E(x) \quad \text{with} \quad E(0) := e \in G. \tag{3.42}$$

Let φ be an arbitrary smooth function on G. Introduce a corresponding function, Φ, on the phase space by

$$\Phi[g] := \varphi(\mathcal{E}). \tag{3.43}$$

From the well-known formula of the variation of $E(x)$, we obtain that

$$A^\Phi(x) = \frac{1}{\kappa}\left(E(x)T^\alpha E^{-1}(x)\right)(\mathcal{R}_\alpha\varphi)(\mathcal{E}) \qquad \text{and} \qquad a^\Phi = 0, \tag{3.44}$$

where $\mathcal{R}_\alpha$ is defined in (3.6). It follows that the conditions imposed by (3.26) and (3.29) are satisfied, actually from (3.29) we get $Y^\Phi(J(x)) = 0$. However, Φ does *not* belong to $\mathtt{H}$ in general. By means of (3.44) we get that

$$A^\Phi_\alpha(0) - A^\Phi_\alpha(2\pi) = \frac{1}{\kappa}\left(\mathcal{D}^-_\alpha\varphi\right)(\mathcal{E}) \qquad (\mathcal{D}^-_\alpha = \mathcal{R}_\alpha - \mathcal{L}_\alpha). \tag{3.45}$$

Because of the condition (3.31), this means that $\Phi \in \mathtt{H}$ precisely if φ is an *invariant* function on G with respect to the adjoint action of $\mathcal{G}$ on G. Examples of invariant functions are furnished by the trace of $\mathcal{E}^k$ $(k = 1, 2, \ldots)$ in some representation. That only the invariant functions of $\mathcal{E}$ are admissible is a well-known result in the context of current algebras, where they provide the Casimir functions of J. In our context, we obtain from the above that for an invariant function φ

$$Y^\Phi(g(x)) = \frac{1}{\kappa}g(x)T^\alpha(\mathcal{R}_\alpha\varphi)(M), \quad Y^\Phi(J(x)) = 0, \quad Y^\Phi(M) = 0. \tag{3.46}$$

To derive these, we used that, since g and E satisfy the same differential equation, $g(x) = E(x)g(0)$. Hence $M = g^{-1}(0)\mathcal{E}g(0)$, and for an invariant function

$$g^{-1}(0)T^\alpha g(0)\left(\mathcal{R}_\alpha\varphi\right)(\mathcal{E}) = T^\alpha\left(\mathcal{R}_\alpha\varphi\right)(M). \tag{3.47}$$

The monodromy matrix M is not a function of J alone, but its invariant functions coincide with those of $\mathcal{E}$, and we have just seen that these functions belong to $\mathtt{H}$. Let us now take an arbitrary smooth function, ψ, on G and associate with it a function, Ψ, on the chiral WZNW phase space by $\Psi[g] := \psi(M)$. Using (3.18) and the definition of the parameter of a vector field, eq. (3.15), one gets that

$$A^\Psi(x) = \frac{1}{\kappa}\left(g(x)T^\alpha g^{-1}(x)\right)(\mathcal{R}_\alpha\psi)(M) \quad \text{and} \quad a^\Psi = \frac{1}{\kappa}T^\alpha(\mathcal{D}^-_\alpha\psi)(M). \tag{3.48}$$

It follows that $\Psi \in \mathtt{H}$. For the hamiltonian vector field we obtain $Y^\Psi(J(x)) = 0$ and

$$g^{-1}(x)Y^\Psi(g(x)) = \frac{1}{2\kappa}T^\alpha\left(\mathcal{D}^+_\alpha\psi\right)(M) + \frac{1}{\kappa}T_\alpha r^{\alpha\beta}(M)\left(\mathcal{D}^-_\beta\psi\right)(M). \tag{3.49}$$

Let us elaborate this for the functions defined by the matrix elements of M in some representation Λ of G. We denote these matrix elements as M^Λ_{ij} and denote by $g^\Lambda_{ij}(x)$ the corresponding matrix element of $g(x)$. Now we shall use $\hat{r}(M)$ in (3.38) and

$$\hat{r}_\pm(M) = \hat{r}(M) \pm \frac{1}{2}\hat{I}, \quad M_1 = M \otimes 1, \quad M_2 = 1 \otimes M. \tag{3.50}$$

Then (3.49) can be rewritten in the tensorial form

$$Y^{M^\Lambda_{kl}}(g^\Lambda_{ij}(x)) = \frac{1}{\kappa}(g(x) \otimes M\,\hat{\Theta}(M))^\Lambda_{ik,jl} \tag{3.51}$$

where

$$\hat{\Theta}(M) := \hat{r}_+(M) - M_2^{-1}\hat{r}_-(M)M_2 \tag{3.52}$$

is taken in the corresponding representation of $\mathcal{G}$, and our notation is $(K \otimes L)_{ik,jl} = K_{ij}L_{kl}$. Furthermore, we obtain

$$Y^{M^\Lambda_{kl}}(M^\Lambda_{ij}) = \frac{1}{\kappa}\Big(M \otimes M\,\hat{\Delta}(M)\Big)^\Lambda_{ik,jl} \tag{3.53}$$

with

$$\hat{\Delta}(M) := \hat{\Theta}(M) - M_1^{-1}\hat{\Theta}(M)M_1. \tag{3.54}$$

We shall comment on the interpretation of these equations later on.

The PB of two smooth functions F_1 and F_2 on a finite dimensional smooth symplectic manifold is defined by the standard formula

$$\{F_1, F_2\} = Y^{F_2}(F_1) = -Y^{F_1}(F_2) = \Omega(Y^{F_2}, Y^{F_1}), \tag{3.55}$$

where Y^{F_i} is the hamiltonian vector field associated with F_i by the symplectic form Ω. The so obtained Poisson algebra is closed under pointwise multiplication of the functions as well as under the PB. One may formally apply the same formula in the infinite dimensional case to the admissible smooth functions that possess a hamiltonian vector field. However, it then may be a very nontrivial problem to fully specify the set of functions that form a closed Poisson algebra, and are a complete set in the sense that they separate the points of the phase space. In our case, it is clear from the foregoing formulae that the products of *local functionals of the current* J and the *smooth functions of the monodromy matrix* M form two subsets of $\mathtt{H}$ that are separately closed under the PB. Moreover, these two subsets commute with each other under the PB (they should clearly be each others centralizer in an appropriate Poisson algebra). But they do not form a complete set of functions on our phase space, since the fundamental field $g(x)$ cannot be completely reconstructed out of $J(x)$ and M.

Let us again consider a representation $\Lambda : G \to GL(V)$ of G. Since the evaluation functions $F^x_{ij}[g] = g^\Lambda_{ij}(x)$ are not elements of $\mathtt{H}$, we smear out the local field and define

$$F_\phi[g] := \int_0^{2\pi} dx \mathrm{Tr}\,\Big(\phi(x)g^\Lambda(x)\Big), \tag{3.56}$$

where $\phi : \mathbf{R} \mapsto \mathrm{End}(V)$ is a smooth test function. It is then easy to see from (3.26) that

$$\begin{aligned} A^{F_\phi}(x) &= \frac{1}{\kappa}g(x)T^\alpha g^{-1}(x)\int_x^{2\pi} dy \mathrm{Tr}\,\Big(\phi(y)g^\Lambda(y)T^\Lambda_\alpha\Big) \quad \text{for} \quad x \in [0, 2\pi], \\ a^{F_\phi} &= \frac{1}{\kappa}T^\alpha\int_0^{2\pi} dy \mathrm{Tr}\,\Big(\phi(y)g^\Lambda(y)T^\Lambda_\alpha\Big). \end{aligned} \tag{3.57}$$

By inspecting the condition that $Y^{F_\phi}(J(x))$ in (3.29) must be periodic, we find that $F_\phi \in \mathtt{H}$ for those ϕ that satisfy

$$\phi^{(k)}(0) = \phi^{(k)}(2\pi) = 0\,, \qquad\qquad k = 0, 1, ... \tag{3.58}$$

Assuming that this holds, the hamiltonian vector field Y^{F_ϕ} is obtained from (3.32) as

$$g^{-1}(x)Y^{F_\phi}(g(x)) = \frac{1}{\kappa}T^\alpha \int_x^{2\pi} dy \mathrm{Tr}\,\left(T^\Lambda_\alpha \phi(y) g^\Lambda(y)\right) - \frac{1}{2}a^{F_\phi} + r(M)(a^{F_\phi}), \quad x \in [0, 2\pi]. \tag{3.59}$$

This permits the following interpretation. Let us define the 'Poisson bracket' of the evaluation functions by the equality

$$Y^{F_\phi}(F_\chi) := \{F_\chi, F_\phi\} := \int_0^{2\pi}\int_0^{2\pi} dxdy \mathrm{Tr}\,_{12}\left(\chi(x)\otimes\phi(y)\left\{g^\Lambda(x) \overset{\otimes}{,} g^\Lambda(y)\right\}\right), \tag{3.60}$$

where $\mathrm{Tr}\,_{12}$ means the normalized trace over $V \otimes V$ and

$$\left\{g^\Lambda(x) \overset{\otimes}{,} g^\Lambda(y)\right\}_{ik,jl} = \left\{g^\Lambda_{ij}(x), g^\Lambda_{kl}(y)\right\}. \tag{3.61}$$

With these definitions, formula (3.59) of the hamiltonian vector field is equivalent to

$$\left\{g^\Lambda(x) \overset{\otimes}{,} g^\Lambda(y)\right\} = \frac{1}{\kappa}\Big(g^\Lambda(x)\otimes g^\Lambda(y)\Big)\Big(\hat{r}(M) + \frac{1}{2}\hat{I}\,\mathrm{sign}\,(y-x)\Big)^\Lambda, \quad 0 < x, y < 2\pi. \tag{3.62}$$

Indeed, upon integration the right hand side of (3.60) equals $Y^{F_\phi}(F_\chi)$ for any functions ϕ and χ subject to (3.58). This equation has the form of a quadratic exchange algebra type PB for the field $g(x)$. Such a classical exchange algebra is usually regarded as a classical analogue of a quantum group symmetry in the chiral WZNW model, but observe that in general our r-matrix is *monodromy dependent*.

The admissible Hamiltonians of type $\mathcal{F}_\mu$, Ψ and F_ϕ that we studied in the above should together generate a closed Poisson algebra. Although at present we cannot fully characterize the set of elements that belong to this algebra, we wish to point out that the Jacobi identity for three functions of type F_ϕ, in any Poisson algebra that contains them, is equivalent to the following equation for $\hat{r}(M)$:

$$[\hat{r}_{12}(M), \hat{r}_{23}(M)] + \Theta_{\alpha\beta}(M)T^\alpha_1\mathcal{R}^\beta\hat{r}_{23}(M) + \text{cycl. perm.} = -\frac{1}{4}\hat{f}. \tag{3.63}$$

Here

$$\hat{f} = f_{\alpha\beta\gamma}T^\alpha\otimes T^\beta\otimes T^\gamma \tag{3.64}$$

and the cyclic permutation is over the three tensorial factors with $\hat{r}_{23} = r^{\alpha\beta}(1\otimes T_\alpha\otimes T_\beta)$, $T^\alpha_1 = T^\alpha\otimes 1\otimes 1$ and so on. Furthermore, we use the components of $\hat{\Theta} = \Theta_{\alpha\beta}T^\alpha\otimes T^\beta$ given by (3.52), for which

$$\Theta_{\alpha\beta}\mathcal{R}^\beta = \frac{1}{2}\mathcal{D}^+_\alpha + r_\alpha{}^\beta\mathcal{D}^-_\beta. \tag{3.65}$$

Eq. (3.63) can be viewed as a dynamical generalization of the classical modified YB equation, to which it reduces if the r-matrix is a monodromy independent constant. Of course, (3.63) is satisfied for any $\hat{r}(M)$ that arises as a solution of (3.34), since the Jacobi identity is guaranteed by $d\Omega^\rho_{chir} = 0$.

For later reference, let us comment here on the analogue of eq. (3.63) that appears in connection with chiral WZNW Bloch waves. The space of Bloch waves[3] in question is defined as

$$\mathcal{M}_{Bloch} := \{ b \in C^\infty(\mathbf{R}, G) \,|\, b(x+2\pi) = b(x)e^\omega, \quad \omega \in \mathcal{A} \subset \mathcal{H} \}, \tag{3.66}$$

where $\mathcal{A}$ is a certain domain in a Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$. There is a natural symplectic form on this space, which is induced by the embedding $\mathcal{M}_{Bloch} \subset \mathcal{M}_{chir}$ and is given by $\kappa\Omega_{Bloch}$ with

$$\Omega_{Bloch}(b) = -\frac{1}{2}\int_0^{2\pi} dx \,\mathrm{Tr}\, \Big(b^{-1}db\Big) \wedge \Big(b^{-1}db\Big)' - \frac{1}{2}\mathrm{Tr}\, \Big((b^{-1}db)(0) \wedge d\omega\Big). \tag{3.67}$$

It is known [8, 9, 10] (for a proof in the spirit of the present paper, see [22]) that the PBs associated with (3.67) are encoded by the following classical exchange algebra:

$$\Big\{ b(x) \overset{\otimes}{,} b(y) \Big\} = \frac{1}{\kappa}\Big(b(x) \otimes b(y)\Big)\Big(\hat{\mathcal{R}}(\omega) + \frac{1}{2}\hat{I}\,\mathrm{sign}\,(y-x)\Big), \quad 0 < x, y < 2\pi, \tag{3.68}$$

$$\hat{\mathcal{R}}(\omega) = \frac{1}{4}\sum_\alpha |\alpha|^2 \coth(\frac{1}{2}\alpha(\omega)) E_\alpha \otimes E_{-\alpha}. \tag{3.69}$$

The domain $\mathcal{A}$ is chosen so that $\alpha(\omega) \notin i2\pi\mathbf{Z}$ for any root α and the root vectors E_α satisfy the normalization $\mathrm{Tr}\,(E_\alpha E_{-\alpha}) = \frac{2}{|\alpha|^2}$. As was first pointed out in [8], the Jacobi identity of the PB in this case gives rise to the equation

$$[\hat{\mathcal{R}}_{12}(\omega), \hat{\mathcal{R}}_{23}(\omega)] + \sum_k H_1^k \frac{\partial}{\partial\omega^k}\hat{\mathcal{R}}_{23}(\omega) + \text{cycl. perm.} = -\frac{1}{4}\hat{f}, \tag{3.70}$$

where $\omega_k = \mathrm{Tr}\,(\omega H_k)$ with respect to a basis H_k of $\mathcal{H}$ whose dual basis is denoted as H^k. The same classical dynamical YB equation appears in other contexts [25, 26, 27], too, and has recently received lot of attention [21, 23, 28, 29].

3.4 Constant exchange r-matrices

We have seen that any symplectic structure $\kappa\Omega^\rho_{chir}$ on $\check{\mathcal{M}}_{chir}$ gives rise to a PB of the form (3.62) governed by an 'exchange r-matrix' $\hat{r}(M)$, which is a solution of eq. (3.63). Those cases for which the exchange r-matrix is M-independent have already been discussed by Falceto-Gawedzki [14] and others. The main point [14] here is that one can construct an appropriate 2-form ρ out of any constant, antisymmetric solution $\hat{r}$ of the modified classical YB equation,

$$[\hat{r}_{12}, \hat{r}_{23}] + [\hat{r}_{13}, \hat{r}_{23}] + [\hat{r}_{12}, \hat{r}_{13}] = -\frac{1}{4}\hat{f}, \tag{3.71}$$

and then *the same* $\hat{r}$ appears in the exchange algebra determined by $\kappa\Omega^\rho_{chir}$. We below present the construction of [14], showing that in our formalism it is easy to give a complete proof as well.

All antisymmetric solutions of (3.71) are known. In fact, Belavin and Drinfeld [30] classified the solutions in the case of a complex simple Lie algebra and their solutions belong also to the normal real form. For other real forms very few solutions survive (see Theorem 3.3 in [31]).

[3] In this context $\mathcal{G}$ is either a complex simple Lie algebra or its normal or compact real form.

There is no solution for the compact real form, because of the negative sign of the coefficient on the right hand side. To explain the mechanism [14] whereby constant exchange r-matrices appear in the chiral WZNW model, we first need to recall a few standard results on eq. (3.71), which can be found in the reviews (e.g. [14, 32]).

In association with a solution of (3.71), $\hat{r} = r^{\alpha\beta} T_\alpha \otimes T_\beta \in \mathcal{G} \wedge \mathcal{G}$, one has the constant linear operators r and $r_\pm = r \pm \frac{1}{2} I$. It follows from (3.71) that the formula

$$[A, B]_r = [r(A), B] + [A, r(B)], \qquad A, B \in \mathcal{G}, \tag{3.72}$$

defines a new Lie bracket on the linear space $\mathcal{G}$; the Lie algebra $(\mathcal{G}, [\ ,\ ]_r)$ is denoted as $\mathcal{G}_r$. Then $r_\pm : \mathcal{G}_r \to \mathcal{G}$ are Lie algebra homomorphisms,

$$[r_\pm(A), r_\pm(B)] = r_\pm\left([A, B]_r\right). \tag{3.73}$$

Any $A \in \mathcal{G}$ can be decomposed as

$$A = A_+ - A_- \qquad \text{with} \qquad A_\pm := r_\pm(A). \tag{3.74}$$

As a consequence of (3.73), one has the following equality of linear operators on $\mathcal{G}$,

$$\exp\left(\mathrm{ad} A_\pm\right) \circ r_\pm = r_\pm \circ \exp\left(\mathrm{ad}_r A\right) \qquad \forall A \in \mathcal{G}. \tag{3.75}$$

Here $\mathrm{ad}A$ and $\mathrm{ad}_r A$ are defined by $(\mathrm{ad}A)(B) = [A, B]$ and $(\mathrm{ad}_r A)(B) = [A, B]_r$ for any $A, B \in \mathcal{G}$. Note also that there exists a neighbourhood of the unit element in G, now denoted as $\check{G} \subset G$, whose elements, $M \in \check{G}$, admit a unique decomposition in the form

$$M = M_+ M_-^{-1} \qquad \text{with} \qquad M_\pm = e^{\Gamma_\pm}, \tag{3.76}$$

where Γ varies in a neighbourhood of zero in $\mathcal{G}$. On $\check{G}$ one has

$$M^{-1} dM = M_- \left(M_+^{-1} dM_+ - M_-^{-1} dM_- \right) M_-^{-1}. \tag{3.77}$$

Let X be an arbitrary vector field on $\check{G}$. By means of (3.73), (3.77) leads to

$$M_\pm^{-1} dM_\pm(X) = \left(M_-^{-1} M^{-1} dM(X) M_- \right)_\pm. \tag{3.78}$$

On the right hand side the subscript refers to the decomposition (3.74).

According to [14], the definition

$$\rho(M) := \frac{1}{2} \mathrm{Tr}\left(M_+^{-1} dM_+ \wedge M_-^{-1} dM_- \right) \tag{3.79}$$

yields a 2-form on $\check{G}$ for which

$$d\rho = \frac{1}{6} \mathrm{Tr}\left(M^{-1} dM \wedge M^{-1} dM \wedge M^{-1} dM \right) \qquad \text{on} \qquad \check{G}. \tag{3.80}$$

It is straightforward to verify (3.80) by using (3.78), (3.73) and the antisymmetry of the r-matrix, which imply e.g. that $\mathrm{Tr}\left(A_+[B_+, C_+]\right) = \mathrm{Tr}\left(A_-[B_-, C_-]\right)$ for any $A, B, C \in \mathcal{G}$.

Coming now to the main point, let us define $\check{\mathcal{M}}_{chir} \subset \mathcal{M}_{chir}$ to be the submanifold where the monodromy matrix is restricted to $\check{G}$. Thanks to (3.80), ρ in (3.79) yields a symplectic form $\kappa\Omega^{\rho}_{chir}$ by (2.27). It is stated in [14] (without a proof) that the PB (3.62) on $\check{\mathcal{M}}_{chir}$ that results is in this case governed by the same constant r-matrix $\hat{r}$ out of which ρ (3.79) has been constructed. Our formalism permits us to prove this important result as follows.

First, we need to rewrite the 2-form ρ in (3.79) in the notation used in (3.20). With the aid of (3.78), we obtain that

$$q^{\alpha\beta}(M) = W^{\alpha\gamma}(M_-) r_{\gamma\theta} W^{\theta\beta}(M_-^{-1}). \tag{3.81}$$

where we employ the notation of (3.3). Here $r_{\alpha\beta}$ refers to the solution of (3.71) that we used to define ρ, and we have to show that this monodromy independent r-matrix also solves the defining equation of the exchange r-matrix, eq. (3.36).

As a consequence of (3.81), for the operators $r_\pm$ and $q_\pm$ that appear in (3.36) we have

$$q_\pm(M) = \mathrm{Ad}\,(M_-) \circ r_\pm \circ \mathrm{Ad}\,(M_-^{-1}). \tag{3.82}$$

By using this together with $M = M_+ M_-^{-1}$ (3.76) and $\mathrm{Ad}\, M_\pm = \exp\,(\mathrm{ad}\, \Gamma_\pm)$, the desired identity (3.36) becomes equivalent to

$$r_+ \circ \exp\,(-\mathrm{ad}\,\Gamma_-) \circ r_- = r_- \circ \exp\,(-\mathrm{ad}\,\Gamma_+) \circ r_+. \tag{3.83}$$

Because of (3.75), the last equation is in turn equivalent to

$$r_+ \circ r_- \circ \exp\,(-\mathrm{ad}_r \Gamma) = r_- \circ r_+ \circ \exp\,(-\mathrm{ad}_r \Gamma)\,, \tag{3.84}$$

which is obviously valid since the operators r_+ and r_- commute. This proves that the constant r-matrix underlying ρ in (3.79) does indeed coincide with the exchange r-matrix associated by (3.36) with the corresponding symplectic form $\kappa\Omega^{\rho}_{chir}$.

A well-known feature of a constant exchange r-matrix is that it naturally admits a Poisson-Lie action of the group G on $\mathcal{M}_{chir}$. Observe that if $\hat{r}$ in (3.62) is independent of M, then the Poisson structure on $\check{\mathcal{M}}_{chir}$ smoothly extends to a Poisson structure on the full $\mathcal{M}_{chir}$. At the same time, one can give G the structure of a PL group by the definition [33, 34]

$$\{h \overset{\otimes}{,} h\} = \frac{1}{\kappa}[\hat{r}, h \otimes h] \qquad h \in G. \tag{3.85}$$

Then one obtains a natural (left) PL action of G on $\mathcal{M}_{chir}$ by the map

$$\mathcal{M}_{chir} \times G \ni (g, h) \mapsto g h^{-1} \in \mathcal{M}_{chir}. \tag{3.86}$$

Indeed, this is a Poisson map if $\mathcal{M}_{chir} \times G$ carries the direct product of the exchange algebra PB on $\mathcal{M}_{chir}$ and the Sklyanin bracket (3.85) on G. In the present case, the meaning of eqs. (3.51)-(3.54) is that $\mathcal{M}_{chir} \ni g \mapsto M = g^{-1}(x) g(x + 2\pi)$ provides the 'non-Abelian' momentum map [14, 32] for this PL action. (Of course, an equivalent right PL action is obtained by replacing h^{-1} with h and using the opposite of the PB on G.)

The above mechanism cannot be used to define a PL symmetry on $\mathcal{M}_{chir}$ in the case of a compact Lie group G, since (3.71) has no solution for a compact $\mathcal{G}$. This is somewhat puzzling since as a quantum field theory the WZNW model is usually considered in the compact domain, where various manifestations of quantum group symmetries were found in the literature [16]. Later we shall see that PL symmetries can be defined on the chiral WZNW phase space by certain mechanisms different from the one described above, and they work in the compact case too.

3.5 A parametrization of $\mathcal{M}_{chir}$ and classical $\mathcal{G}$-symmetry

We below introduce local coordinates on the chiral WZNW phase space consisting of a periodic G-valued field and the logarithm, Γ, of the monodromy matrix. This will lead us to realize the existence of a special choice of ρ such that with respect to $\kappa\Omega^{\rho}_{chir}$ Γ generates an infinitesimal symplectic action of $\mathcal{G}$ on $\check{\mathcal{M}}_{chir}$, i.e., a classical $\mathcal{G}$-symmetry. The parametrization will also shed a new light on the notion of admissible Hamiltonians.

If the monodromy matrix M is near to $e \in G$, then the chiral WZNW field can be uniquely parametrized as

$$g(x) = h(x)e^{x\Gamma}, \tag{3.87}$$

where $h(x)$ is a G-valued, smooth, 2π-periodic function and Γ varies in a neighbourhood of zero in $\mathcal{G}$, $\check{\mathcal{G}} \subset \mathcal{G}$, for which the map $\check{\mathcal{G}} \ni \Gamma \mapsto M = e^{2\pi\Gamma} \in \check{G} \subset G$ is a diffeomorphism. We may identify a domain in $\mathcal{M}_{chir}$ with the corresponding space of parameters,

$$\check{\mathcal{M}}_{chir} = \widetilde{G} \times \check{\mathcal{G}} = \{(h, \Gamma)\}. \tag{3.88}$$

An easy computation gives the following formula for Ω_{chir} (2.17) in this parametrization:

$$\Omega_{chir}(h, \Gamma) = \Omega^0_{chir}(h, \Gamma) - \rho_0(\Gamma), \tag{3.89}$$

where

$$\Omega^0_{chir}(h, \Gamma) = -\frac{1}{2}\int_0^{2\pi} dx \,\mathrm{Tr}\,\left(h^{-1}dh \wedge (h^{-1}dh)'\right) + d\int_0^{2\pi} dx \mathrm{Tr}\,\left(\Gamma h^{-1}dh\right), \tag{3.90}$$

$$\rho_0(\Gamma) = -\frac{1}{2}\int_0^{2\pi} dx\,\mathrm{Tr}\,\left(d\Gamma \wedge de^{x\Gamma}e^{-x\Gamma}\right). \tag{3.91}$$

Taking into account that $M = e^{2\pi\Gamma}$, it is not difficult to verify that

$$d\rho_0(\Gamma) = \frac{1}{6}\mathrm{Tr}\,\left(M^{-1}dM \wedge M^{-1}dM \wedge M^{-1}dM\right). \tag{3.92}$$

Upon comparison with (2.18), this implies that $d\Omega^0_{chir} = 0$, which one can check directly as well. Recalling eq. (2.28), we then notice that the 2-form ρ in (2.27) in this case can be parametrized by an arbitrary closed 2-form β on $\check{\mathcal{G}}$ as

$$\rho(\Gamma) = \rho_0(\Gamma) + \beta(\Gamma), \qquad d\beta(\Gamma) = 0. \tag{3.93}$$

By (2.27) we thus have $\Omega^{\rho}_{chir} = \Omega^0_{chir} + \beta$, in particular $\Omega^{\rho_0}_{chir} = \Omega^0_{chir}$. In order to determine the exchange r-matrix, $\hat{r}_0$, corresponding to ρ_0, we note that the integral defining ρ_0 can be computed in closed form. In fact, the linear operator, q_0, associated with its matrix in (3.21) according to (3.2) turns out to be[4]

$$q_0 = \frac{2\mathcal{Y} + e^{-\mathcal{Y}} - e^{\mathcal{Y}}}{2(e^{\mathcal{Y}} - 1)(1 - e^{-\mathcal{Y}})} \qquad \text{with} \qquad \mathcal{Y} := 2\pi(\mathrm{ad}\,\Gamma). \tag{3.94}$$

[4]The expressions in eqs. (3.94), (3.95) are defined by the power series expansions of the corresponding complex analytic functions around zero. For instance [35], $2r_0 = \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{(2k)!}(\frac{1}{2}\mathcal{Y})^{2k-1}$.

Then from eq. (3.36) we find the linear operator version, r_0, of the exchange r-matrix as

$$r_0 = \frac{1}{2}\coth\frac{\mathcal{Y}}{2} - \frac{1}{\mathcal{Y}}\,. \tag{3.95}$$

By means of (3.62) this r-matrix defines one of the possible monodromy dependent exchange algebras for the chiral WZWN field, and it also represents a nontrivial solution of (3.63). In the knowledge of the r-matrix, the PBs containing M can be determined straightforwardly from (3.51), (3.53). In particular, it is easy to see that the hamiltonian vector field generated by the function $\Gamma_\alpha = \mathrm{Tr}\,(T_\alpha\Gamma)$ gives rise to the PBs

$$\{g(x), \tilde{\Gamma}_\alpha\} = -g(x)T_\alpha \quad \text{and} \quad \{\tilde{\Gamma}_\alpha, \tilde{\Gamma}_\beta\} = f^{\gamma}_{\alpha\beta}\tilde{\Gamma}_\gamma \quad \text{for} \quad \tilde{\Gamma}_\alpha := -2\pi\kappa\Gamma_\alpha\,. \tag{3.96}$$

This means that in the case of the symplectic form $\kappa\Omega^0_{chir}$ the logarithm of M generates a *classical* $\mathcal{G}$-symmetry on $\check{\mathcal{M}}_{chir}$. Indeed, the momentum map corresponding to this symmetry is just $\tilde{\Gamma}$. A classical $\mathcal{G}$-symmetry is sometimes called 'Abelian' to contrast it with a proper ('non-Abelian') PL symmetry, for which the symmetry group itself is endowed with a nonzero PB [32].

The above construction is valid for any simple Lie group. Perhaps even more surprisingly than the possibility to define a classical $\mathcal{G}$-symmetry on $\check{\mathcal{M}}_{chir}$ for any $\mathcal{G}$, in sec. 5 it will turn out that the symplectic structure on $\check{\mathcal{M}}_{chir}$ can be chosen so as to be compatible with any prescribed PL structure on G.

Now we explain how the characterization of the admissible Hamiltonians found in sec. 3.2 appears in the coordinates (h, Γ). For this, we first remark that on account of (3.88) it is natural to represent a vector field X on $\check{\mathcal{M}}_{chir}$ as

$$X = (X_h, X_\Gamma) \qquad X_h \in T_h\widetilde{G} \qquad X_\Gamma \in \mathcal{G} \tag{3.97}$$

with $h^{-1}X_h \in \widetilde{\mathcal{G}}$. By regarding h and Γ as evaluation functions, we may write $X_h = X(h)$ and $X_\Gamma = X(\Gamma)$. Of course, the derivative $X(F)$ of function F on $\check{\mathcal{M}}_{chir}$ with respect to X is still defined by means of a curve to which X is tangent. Let us call a function F *periodically differentiable* if its derivative with respect to any X exists and has the form

$$X(F) = \langle dF, X\rangle = \mathrm{Tr}\,(d_\Gamma F X_\Gamma) + \int_0^{2\pi} \mathrm{Tr}\,\Big((h^{-1}d_hF)(h^{-1}X_h)\Big) \tag{3.98}$$

with the exterior derivative

$$dF = (d_hF, d_\Gamma F) \qquad d_hF \in T^*_h\widetilde{G} \qquad d_\Gamma F \in \mathcal{G}, \tag{3.99}$$

where $h^{-1}d_hF \in \widetilde{\mathcal{G}}$ by the natural identifications. Our point then is that the periodically differentiable functions coincide precisely with the admissible Hamiltonians. To prove this, recall that the definition of an admissible Hamiltonian was that its derivative exists with respect to any vector field and has the form (3.26), where A^F assigns to any $g \in \check{\mathcal{M}}_{chir}$ a smooth $\mathcal{G}$-valued function on $\mathbf{R}$ such that (3.29) gives a 2π-periodic function, and a^F is given by (3.31). If we now suppose that the derivative of a function F has the form in (3.26), where $A^F(x)$ is a smooth function on $\mathbf{R}$, then by inserting the parametrization (3.87) and performing a few partial integrations we obtain

$$
\begin{aligned}
X(F) = &\int_0^{2\pi} dx\, \mathrm{Tr}\, \left(X(\kappa g' g^{-1})A^F\right) + \kappa \mathrm{Tr}\, \left(g^{-1}(0) X(g(0)) a^F\right) \\
= &\int_0^{2\pi} dx\, \mathrm{Tr}\, \left(\left(X(h)h^{-1}\right)\left(-\kappa (A^F)' - [A^F, \kappa g' g^{-1}]\right)\right) \\
&+ \kappa \mathrm{Tr}\, \left(X(\Gamma)\int_0^{2\pi} dx\, (g^{-1}A^F g)\right) + \kappa \int_0^{2\pi} dx\, \mathrm{Tr}\, \left([e^{-x\Gamma}X(e^{x\Gamma}), \Gamma](g^{-1}(x)A^F(x)g(x))\right) \\
&+ \kappa \mathrm{Tr}\, \left(\left(g^{-1}(0)Xg(0)\right)\left(a^F + g^{-1}(0)[A^F(2\pi) - A^F(0)]g(0)\right)\right) \qquad (3.100)
\end{aligned}
$$

for any vector field X on $\check{\mathcal{M}}_{chir}$. Clearly, this formula can be rewritten in the form of (3.98) if and only if

$$
a^F + g^{-1}(0)[A^F(2\pi) - A^F(0)]g(0) = 0, \qquad (3.101)
$$

$$
-(d_h F)h^{-1} = \kappa (A^F)' + [A^F, \kappa g' g^{-1}] \qquad (3.102)
$$

and

$$
\begin{aligned}
\mathrm{Tr}\,(X(\Gamma) d_\Gamma F) &= \kappa \int_0^{2\pi} dx\, \mathrm{Tr}\, \left(\left(X(\Gamma) + [e^{-x\Gamma}X(e^{x\Gamma}), \Gamma]\right)(g^{-1}A^F g)\right) \\
&= \kappa \int_0^{2\pi} dx\, \mathrm{Tr}\, \left(\left(e^{-x\Gamma}X(e^{x\Gamma})\right)'(g^{-1}A^F g)(x)\right) \\
&= \kappa \mathrm{Tr}\, \left(X(\Gamma)\int_0^{2\pi} dx\, (h^{-1}A^F h)(x)\right). \qquad (3.103)
\end{aligned}
$$

The last equality follows by an easy calculation, and it implies that

$$
d_\Gamma F = \kappa \int_0^{2\pi} (h^{-1}A^F h)(x). \qquad (3.104)
$$

We conclude that a function F for which $X(F)$ has the form of the first line in (3.100) is periodically differentiable if and only if (3.101) is satisfied and the right hand side of (3.102) defines a smooth, 2π-periodic function. In particular, all admissible Hamiltonians are periodically differentiable. Conversely, every periodically differentiable function is admissible in the sense of sec. 3.2 since it is possible to uniquely determine (A^F, a^F) in terms of $(d_h F, d_\Gamma F)$ in such a way (3.98) is converted into the first line of (3.100). To achieve this, if $(d_h F, d_\Gamma F)$ are given, one has to solve the differential equation (3.102) together with the condition in (3.104) for $A^F(x)$, and then one may define a^F by (3.101). It is not difficult to show that, since Γ is restricted to $\check{\mathcal{G}}$, (3.104) has a unique solution for the initial value $A^F(0)$ of $A^F(x)$, which completes the proof.

4 Kinematical derivation of the Poisson brackets

In this section we rederive the PBs of the chirally extended WZNW model using purely 'kinematical' considerations. Instead of explicitly inverting the symplectic form, we postulate the natural properties of the chiral extension (which we have established in the symplectic formalism) and this way we can reproduce the quadratic exchange algebras (3.62) and (3.68) almost effortlessly. This is especially so in the case of Bloch waves, where the dynamical r-matrix (3.69) is determined algebraically. The subsequent considerations are complementary to the symplectic approach presented in sec. 3 and, for Bloch waves, in [22]. The kinematical derivation sheds a new light on the origin of the chiral exchange algebras, and some issues are also easier to discuss in this approach.

We have seen in sec. 2 that it is very natural to extend the WZNW phase space as

$$\mathcal{M} \approx \mathcal{M}^{sol} \quad \rightarrow \quad \mathcal{M}^{ext} = \check{\mathcal{M}}_L \times \check{\mathcal{M}}_R \,, \tag{4.1}$$

where $\check{\mathcal{M}}_L$ and $\check{\mathcal{M}}_R$ are two identical copies of the chiral phase space characterized by the smooth, quasiperiodic chiral fields $g_C(x+2\pi) = g_C(x)\, M_C\,,\ C = L, R$. Since the separation of the chiral degrees of freedom is an essential feature of the WZNW model, we shall assume that $\check{\mathcal{M}}_L$ and $\check{\mathcal{M}}_R$ are independent and they are equipped with the same symplectic structure (up to an overall sign difference, see eq. (2.16)). The corresponding Poisson algebra will be supposed to contain the important Hamiltonians $\mathcal{F}_\mu$, Ψ and F_ϕ studied in sec. 3.3. The further main assumptions of the kinematical approach are that the constraints $M_L = M_R$ are first class and the corresponding gauge transformations[5] operate according to (2.10) so that the WZNW solution $g(\sigma,\tau) = g_L(x_L)\, g_R^{-1}(x_R)$ is gauge invariant. These assumptions, together with simple properties of the original WZNW phase space $\mathcal{M}^{sol}$, allow us to reproduce the PBs (3.62) and (3.68).

4.1 The chiral Poisson brackets for generic monodromy

From now on we mainly concentrate on the chiral half of the problem and, for notational simplicity, omit the subscript C, wherever it is possible.

We start by noting that because the Fourier components of the left KM current belong to the space of admissible Hamiltonians, acting on the left chiral field they must generate the transformation

$$\left\{ J_T^n, g(x) \right\} = -T\, e^{-inx} g(x)\,, \qquad \text{where} \qquad J_T^n := \int_0^{2\pi} dx\, e^{-inx} \mathrm{Tr}\,(T J(x))\,, \tag{4.2}$$

which means that the chiral field is a KM primary field. This crucial relation can be obtained by first noting that on the submanifold of $\mathcal{M}^{ext}$ defined by imposing the constraint $M_L = M_R$ (2.21) holds for the product (2.8) that gives the WZNW solution. The gauge invariance of the solution field and the fact that the different chiral pieces completely Poisson commute then allow us to derive (4.2). Of course, an analogous relation is valid for the right-moving fields.

An other quantity which, by assumption, belongs to the space of admissible Hamiltonians is the monodromy matrix M. From (4.2) it follows that

$$\left\{ J_T^n, M \right\} = 0\,, \tag{4.3}$$

which is obvious because M is invariant under the KM transformations. For later use we note that a quantity which Poisson commutes with the Fourier components of the KM current

[5]In the rest of the paper, when considering gauge transformations or G-symmetries, we shall always assume that the domain of allowed monodromy matrices, $\check{G} \subset G$, consists of full conjugacy classes in G. If this did not hold, everything would still be true for *infinitesimal* gauge transformations or $\mathcal{G}$-symmetries.

must be a function of the monodromy matrix M. (This is most easily seen by using the parametrization (3.87).)

We wish to determine the PBs of the 'smeared out' field[6]

$$F_\phi = \int_0^{2\pi} dx \mathrm{Tr}\, (\phi(x)g(x)) \,, \tag{4.4}$$

where the matrix valued test function $\phi(x)$ satisfies (3.58). Note that while in the symplectic approach of sec. 3 the fact that F_ϕ is an admissible Hamiltonian follows from the properties of the symplectic form, here it is an additional assumption. In order to compute the PBs with F_ϕ it is enough to find

$$B_\phi(x) := \Big\{F_\phi, g(x)\Big\} \tag{4.5}$$

for $0 \leq x \leq 2\pi$ ($B_\phi(x)$ corresponds to $-Y^{F_\phi}(g(x))$ in sec. 3.3). To constrain $B_\phi(x)$ we apply (4.2) to F_ϕ and then using the fact that F_ϕ is an admissible Hamiltonian we obtain the local form

$$\Big\{F_\phi, J(x)\Big\} = \mathrm{Tr}\,\Big(\phi(x)T^\alpha g(x)\Big)T_\alpha \,. \tag{4.6}$$

Comparing (4.5) and (4.6) leads to the differential equation

$$\kappa B'_\phi(x) - J(x)B_\phi(x) = \mathrm{Tr}\,\Big(\phi(x)T^\alpha g(x)\Big)T_\alpha g(x)\,, \tag{4.7}$$

whose solution is

$$B_\phi(x) = \frac{1}{2\kappa}\int_0^{2\pi} dy\,\mathrm{sign}(x-y)\mathrm{Tr}\,\Big(\phi(y)g(y)T^\alpha\Big)g(x)T_\alpha + g(x)U_\phi\,, \tag{4.8}$$

where

$$U_\phi = U_\phi^\alpha T_\alpha \tag{4.9}$$

is a constant element of the Lie algebra. If we consider $F_{\tilde\phi}$, an other Hamiltonian of type F_ϕ belonging to a smearing function $\tilde\phi(x)$, and use the antisymmetry of the PB $\{F_\phi, F_{\tilde\phi}\}$, we obtain

$$U_\phi^\alpha \tilde\phi_\alpha = -U_{\tilde\phi}^\alpha \phi_\alpha\,, \tag{4.10}$$

where

$$\phi_\alpha := \int_0^{2\pi} dx \mathrm{Tr}\,\Big(\phi(x)g(x)T_\alpha\Big) \tag{4.11}$$

and $\tilde\phi_\alpha$ is defined analogously. Eq. (4.10) implies that U_ϕ^α is a linear combination of the integrals ϕ_α of the form

$$U_\phi^\alpha = -\frac{1}{\kappa}\, r^{\alpha\beta}\phi_\beta\,, \tag{4.12}$$

where $r^{\alpha\beta} = -r^{\beta\alpha}$. By means of (4.12), (4.8) becomes equivalent to the classical exchange algebra

$$\Big\{g(x) \overset{\otimes}{,} g(y)\Big\} = \frac{1}{\kappa} g(x)\otimes g(y)\Big(\hat r + \frac{1}{2}\hat I\,\mathrm{sign}(y-x)\Big), \qquad 0 < x,y < 2\pi\,, \tag{4.13}$$

where the r-matrix $\hat r = r^{\alpha\beta}T_\alpha\otimes T_\beta$ is an x-independent constant. Of course, as before, (4.13) has to be interpreted in the distributional sense.

[6]Here some representation Λ of G is used like in (3.56), but henceforth Λ is omitted from all notations.

Although x-independent, the r-matrix can still depend on the phase space. This latter dependence can be restricted by Poisson commuting J_T^n with (4.13) and applying the Jacobi identity. In this way we get

$$\Big\{J_T^n, r^{\alpha\beta}\Big\} = 0\,, \tag{4.14}$$

which implies, as explained earlier, that $\hat{r}$ must be a function of M alone.

Next we consider the PBs of the monodromy matrix. Using (4.5), (4.8) and (4.12) we get

$$\Big\{F_\phi, M\Big\} = g^{-1}(x)\Big(B_\phi(2\pi) - B_\phi(0)M\Big) = \frac{1}{\kappa}\phi_\alpha\left(\frac{1}{2}(MT^\alpha + T^\alpha M) + r^{\alpha\beta}(MT_\beta - T_\beta M)\right) \tag{4.15}$$

and because M is an admissible Hamiltonian this implies the local form

$$\Big\{g(x) \overset{\otimes}{,} M\Big\} = \frac{1}{\kappa} g(x) \otimes M\,\hat{\Theta}(M)\,, \tag{4.16}$$

where $\hat{\Theta}$ is given by (3.52). We obtain in a similar way that

$$\Big\{M \overset{\otimes}{,} M\Big\} = \frac{1}{\kappa} M \otimes M\,\hat{\Delta}(M) \tag{4.17}$$

with $\hat{\Delta}$ defined in (3.54).

To ascertain that our construction is self-consistent, we now show that

$$\mathcal{C} = M_L - M_R \approx 0 \tag{4.18}$$

is a first class constraint on $\mathcal{M}^{ext}$ and the WZNW solution is gauge invariant. In fact, $\mathcal{C}$ is first class since

$$\begin{aligned}
\Big\{\mathcal{C} \overset{\otimes}{,} \mathcal{C}\Big\} &= \Big\{M_L \overset{\otimes}{,} M_L\Big\} + \Big\{M_R \overset{\otimes}{,} M_R\Big\} \\
&= \frac{1}{\kappa}(M_L \otimes M_R\Big)\hat{\Delta}(M_L) - \frac{1}{\kappa}\Big(M_R \otimes M_R\Big)\hat{\Delta}(M_R) \approx 0\,.
\end{aligned}$$

Similarly, the gauge invariance of $g(\sigma,\tau)$ in (2.8) can be shown as follows:

$$\begin{aligned}
\kappa\Big\{g(\sigma,\tau) \overset{\otimes}{,} \mathcal{C}\Big\} &= \Big(g_L(x_L) \otimes M_L\Big)\hat{\Theta}(M_L)\Big(g_R^{-1}(x_R) \otimes 1\Big) \\
&- \Big(g_L(x_L) \otimes M_R\Big)\hat{\Theta}(M_R)\Big(g_R^{-1}(x_R) \otimes 1\Big) \\
&\approx \Big(g_L(x_L) \otimes M\Big)\Big(\hat{\Theta}(M_L) - \hat{\Theta}(M_R)\Big)\Big(g_R^{-1}(x_R) \otimes 1\Big) \approx 0\,.
\end{aligned}$$

Here the notation $\approx$ indicates 'weak' equality, i.e., equality on the constrained manifold, and we used the assumption that $\check{\mathcal{M}}_L$ and $\check{\mathcal{M}}_R$ are independent and carry opposite PBs.

For later use we mention an additional consistency check. Consider the path ordered integral $\mathcal{E} = E(2\pi)$ defined in (3.42). Since $\mathcal{E}$ and M are related by conjugation, their invariants coincide:

$$\varepsilon_N = \mathrm{Tr}\,\left(\mathcal{E}^N\right) = \mathrm{Tr}\,\left(M^N\right) = m_N, \tag{4.19}$$

and since the PBs of the Hamiltonians ε_N can be calculated unambiguously using the KM algebra only, the following relation must hold:

$$\Big\{g(x), m_N\Big\} = \Big\{g(x), \varepsilon_N\Big\} = \frac{N}{\kappa} g(x) T^\alpha \mathrm{Tr}\,\left(M^N T_\alpha\right). \tag{4.20}$$

It is easy to verify by using (3.52) in (4.16) that (4.20) is indeed satisfied.

To summarize, by postulating the $M_L = M_R$ constraint to be first class, as well as the admissible Hamiltonian nature of the Fourier components of the current, the smeared out chiral field and the monodromy matrix, we have established that the possible extensions of the WZNW phase space correspond to the quadratic exchange algebra (4.13) with some monodromy dependent exchange r-matrix. Of course the classical exchange algebra can only provide a valid PB if it satisfies the Jacobi identity. This is guaranteed, by construction, if the r-matrix is obtained as a solution of (3.36). In the present approach, we have to *impose* the Jacobi condition and, as mentioned in sec. 3.3, this leads to the dynamical YB equation (3.63). The chiral extensions of the WZNW model are thus characterized by the solutions of this equation.

Most known solutions of (3.63) are local in that they are defined only on a proper, open submanifold $\check{G} \subset G$. This is obviously the case for the solutions obtained by solving (3.36) for the r-matrix, starting from a q-matrix representing by (3.21) a local solution of (2.28). On the other hand, as the example of constant r-matrices shows, there are global solutions as well. Since constant solutions exist for non-compact groups only, an interesting open question is whether there exist globally defined exchange r-matrices also for compact groups. We hope to return to this question in a future study.

We end this subsection by discussing a generalization of the 'gauge' freedom (2.10) of the chiral extension. It is clear that the gauge transformed chiral fields given by

$$g(x) \;\rightarrow\; (\mathcal{P}g)(x) = g(x)\, p(M) \tag{4.21}$$

with an arbitrary *monodromy dependent* group element $p(M)$ correspond to the same point in the physical phase space after the projection (2.11), provided we apply the same gauge transformation to both chiral fields. In other words, both the original fields, $g(x)$, and the gauge transformed fields, $\tilde{g}(x) := g(x)\, p(M)$, are smooth quasiperiodic G-valued 'coordinates' of the same point in the physical phase space.

The multiplication law for two elements, $\mathcal{P}_1, \mathcal{P}_2$, of this huge gauge group is given by

$$p_{12}(M) = p_2(M)\, p_1(\mathcal{P}_2 M)\,, \tag{4.22}$$

where p_1, p_2 and p_{12} correspond to $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_{12} := \mathcal{P}_1\mathcal{P}_2 := \mathcal{P}_1 \circ \mathcal{P}_2$, respectively. Here

$$\mathcal{P}M := p^{-1}(M) M p(M) \tag{4.23}$$

is how the monodromy matrix itself is transformed under a gauge transformation. To qualify as an element of the gauge group we must also require that the inverse transformation exists. This is equivalent to requiring that the group valued function, $\bar{p}(M)$, corresponding to the inverse element, $\mathcal{P}^{-1}$, exists and solves

$$p(M)\bar{p}(\mathcal{P}M) = e\,. \tag{4.24}$$

In terms of the new 'coordinates' $\tilde{g}(x)$ defined by (4.21) the exchange algebra has the same form as (4.13) with a gauge transformed exchange r-matrix, $\hat{\tilde{r}}$. On account of the Leibniz rule of the PB, one finds that

$$\hat{\tilde{r}} = p^{-1}(M) \otimes p^{-1}(M)\Big(\hat{r}(M) + \Theta^{\alpha\beta}[T_\alpha \otimes \mathcal{A}_\beta - \mathcal{A}_\beta \otimes T_\alpha] + \Delta^{\alpha\beta}\mathcal{A}_\alpha \otimes \mathcal{A}_\beta\Big) p(M) \otimes p(M)\,, \tag{4.25}$$

where $\mathcal{A}_\alpha := (\mathcal{R}_\alpha p)p^{-1}$ and $\hat{\tilde{r}}$ should be expressed as a function of the gauge transformed monodromy $\tilde{M} := \mathcal{P}M$. Since the Jacobi identity of the exchange algebra (4.13) is independent of the coordinates used, it is clear that the solutions of the dynamical YB equation (3.63) are transformed into each other by the elements of the gauge group and therefore can be classified up to gauge transformations.

4.2 Diagonal monodromy

Below we briefly outline a kinematical derivation of the PBs on the space of chiral WZNW Bloch waves, $\mathcal{M}_{Bloch}$ defined in (3.66). In order to emphasize their diagonality, we denote the monodromy matrices of the Bloch waves, $b(x)$, here by D,

$$D = e^{\omega^k H_k}\,, \tag{4.26}$$

where H_k are the basis elements of a splitting Cartan subalgebra of $\mathcal{G}$. We will also use the derivatives $\partial_k = \frac{\partial}{\partial\omega^k}$.

The assumptions and the main steps of the construction (with obvious modifications for the diagonal case) are the same as discussed above for the general case. Now we obtain a classical exchange algebra of the form

$$\Big\{b(x) \stackrel{\otimes}{,} b(y)\Big\} = \frac{1}{\kappa} b(x)\otimes b(y)\Big(\hat{r}(\omega) + \frac{1}{2}\hat{I}\,\mathrm{sign}(y-x)\Big), \qquad 0<x,y<2\pi. \tag{4.27}$$

The main difference comes from requiring (4.20), because, unlike in the general case where it was a consequence of the exchange algebra, here it completely determines the PBs of the monodromy matrix:

$$\Big\{b(x) \stackrel{\otimes}{,} D\Big\} = \frac{1}{\kappa}(b(x)\otimes D)\,(H_k\otimes H^k)\,. \tag{4.28}$$

On the other hand, the analogue of (4.16) implied by (4.27) now reads explicitly as

$$\Big\{b(x) \stackrel{\otimes}{,} D\Big\} = \frac{1}{\kappa}\,(b(x)\otimes D)\,\Big(\hat{r}(\omega) + \frac{1}{2}\hat{I} - (1\otimes D^{-1})(\hat{r}(\omega) - \frac{1}{2}\hat{I})(1\otimes D)\Big). \tag{4.29}$$

The comparison of the last two equations fixes the exchange r-matrix almost completely:

$$\hat{r}(\omega) = \hat{\mathcal{R}}(\omega) + \hat{X}(\omega)\,, \tag{4.30}$$

where $\hat{\mathcal{R}}(\omega)$ is given by (3.69) and $\hat{X}(\omega)$ is an antisymmetric purely Cartan piece,

$$\hat{X}(\omega) = X^{kl}(\omega)H_k\otimes H_l\,, \qquad\qquad X^{kl}(\omega) = -X^{lk}(\omega)\,. \tag{4.31}$$

Thus we have determined the exchange r-matrix algebraically up to the Cartan piece.

The Jacobi identity takes the following form for the diagonal case:

$$[\hat{r}_{12}(\omega), \hat{r}_{23}(\omega)] + H_1^k\,\partial_k\hat{r}_{23}(\omega) + \text{cycl. perm.} = -\frac{1}{4}\hat{f}. \tag{4.32}$$

This is the celebrated CDYB equation, whose neutral solutions have been classified in [23]. The r-matrix (4.30) satisfies (4.32) if

$$\partial_k X_{lm} + \partial_l X_{mk} + \partial_m X_{kl} = 0. \tag{4.33}$$

Therefore there exists a 'gauge vector' $V_k(\omega)$ such that

$$X_{kl}(\omega) = \partial_k V_l(\omega) - \partial_l V_k(\omega). \tag{4.34}$$

With the help of $V_k(\omega)$ we can introduce the gauge transformed chiral field

$$\tilde{b}(x) = b(x) e^{-V_k(\omega) H^k}, \tag{4.35}$$

which has diagonal monodromy and satisfies the 'standard' exchange algebra

$$\left\{\tilde{b}(x) \overset{\otimes}{,} \tilde{b}(y)\right\} = \frac{1}{\kappa} \tilde{b}(x) \otimes \tilde{b}(y) \Big(\hat{\mathcal{R}}(\omega) + \frac{1}{2} \hat{I} \operatorname{sign}(y-x)\Big), \qquad 0 < x, y < 2\pi\,. \tag{4.36}$$

It has been mentioned that the PB (4.36) follows [22] from the symplectic form $\kappa\Omega_{Bloch}(\tilde{b})$ on $\mathcal{M}_{Bloch}$ given by (3.67). Upon the substitution (4.35), the symplectic form gets shifted by the exact 2-form $\kappa X_{kl}(\omega)\, d\omega^k \wedge d\omega^l$ and the shifted symplectic form corresponds of course to the exchange algebra (4.27) with $\hat{r}(\omega)$ in (4.30). It is also easy to see that the family of the symplectic forms $\kappa \left(\Omega_{Bloch}(b) + X_{kl}(\omega)\omega^k \wedge \omega^l\right)$ on $\mathcal{M}_{Bloch}$ results as the reduction of the family of symplectic forms $\kappa\Omega^{\rho}_{chir}$ to diagonal monodromy.

5 Exchange r-matrices with Poisson-Lie symmetry

We studied in sec. 3.4 the Poisson-Lie action of the group G on the chiral phase space for the special case of a constant r-matrix playing the rôle of the exchange r-matrix. As mentioned there, this PL action is restricted to the case of complex or real, non-compact groups, since there is no constant solution of (3.63) for compact groups. In this section we consider a set of more general PL actions which also work for the physically most interesting case of compact groups.

It is clear from the examples studied so far that the PL action we are looking for is a kind of 'hidden' symmetry, extending and centralizing the Kac-Moody symmetries in the total symmetry group of the chiral WZNW model. More precisely, we require that

- the KM currents are invariant under the PL action;
- the PL action commutes with the KM transformations.

It is not difficult to see that because of the above two requirements the PL action as a nonlinear group action on the chiral phase space has to be a special case of the gauge transformations discussed in sec. 4.1:

$$g(x) \ \rightarrow\ \mathcal{P}_h g(x) = g(x) p(M, h) \qquad \forall h \in G. \tag{5.1}$$

The G-valued function $p(M,h)$ is chosen so that the group multiplication law $\mathcal{P}_h \mathcal{P}_k = \mathcal{P}_{hk}$ is satisfied:

$$p(M,k) p(\mathcal{P}_k M, h) = p(M, hk)\,, \tag{5.2}$$

where the induced action on the monodromy matrix is $\mathcal{P}_k M = p^{-1}(M,k) M p(M,k)$. One must also require that $p(M,e) = e$.

The simplest case is the 'standard' (left) action

$$\mathcal{S}_h g(x) = g(x)h^{-1} \tag{5.3}$$

corresponding to $p(M,h) = h^{-1}$. A family of 'trivial' actions is obtained by conjugating the standard action in the gauge group by an arbitrary element $\mathcal{P}$ (see (4.21)):

$$\tilde{\mathcal{S}}_h = \mathcal{P}^{-1}\mathcal{S}_h\mathcal{P}. \tag{5.4}$$

In terms of the corresponding G-valued functions we have

$$\tilde{s}(M,h) = p(M)h^{-1}\bar{p}(h\cdot \mathcal{P}M\cdot h^{-1})\,. \tag{5.5}$$

In practice it is useful to consider the infinitesimal version of (5.1). For the parametrization $h = e^{u^\alpha T_\alpha}$ we have

$$\mathcal{P}_h g(x) = g(x) - u^\alpha X_\alpha g(x) + \mathcal{O}(u^2)\,, \tag{5.6}$$

where the infinitesimal generators are of the form

$$X_\alpha g(x) = -\zeta^\beta{}_\alpha(M)g(x)T_\beta \tag{5.7}$$

with some monodromy dependent coefficients $\zeta^\beta{}_\alpha(M)$, and satisfy the commutation relations $[X_\alpha, X_\beta] = f_{\alpha\beta}{}^\gamma X_\gamma$. This latter equation can be expressed as a requirement on the coefficients $\zeta^\beta{}_\alpha$ as

$$\zeta^\lambda{}_\alpha\mathcal{D}^-_\lambda\zeta^\omega{}_\beta - \zeta^\lambda{}_\beta\mathcal{D}^-_\lambda\zeta^\omega{}_\alpha + f_{\kappa\lambda}{}^\omega\zeta^\kappa{}_\alpha\zeta^\lambda{}_\beta + f_{\alpha\beta}{}^\lambda\zeta^\omega{}_\lambda = 0\,. \tag{5.8}$$

Clearly the simplest solution of (5.8) is the standard one, $\zeta^\beta{}_\alpha = -\delta^\beta{}_\alpha$.

The infinitesimal generators are also useful in studying the question of trivialization. The infinitesimal form of (5.5) is

$$pT_\alpha + \zeta^\lambda{}_\alpha(\mathcal{D}^-_\lambda + T_\lambda)p = 0\,. \tag{5.9}$$

It is easy to see that the consistency conditions of this set of partial differential equations are precisely the equations (5.8), but it is not clear if all possible nonlinear actions $\mathcal{P}_h$ can be trivialized in the form (5.4) or not.

The next question is which of the above nonlinear group actions are Poisson-Lie? Following [32], we recall that a Lie group (or algebra) action characterized by infinitesimal generators X_α is Poisson-Lie, if for any pair of phase space functions F_1, F_2 the PBs satisfy the relations

$$\Big\{X_\alpha(F_1), F_2\Big\} + \Big\{F_1, X_\alpha(F_2)\Big\} - X_\alpha\Big(\{F_1,F_2\}\Big) = -\tilde{f}^{\beta\gamma}{}_\alpha\, X_\beta(F_1)\, X_\gamma(F_2)\,, \tag{5.10}$$

where the pair of structure constants $(f_{\alpha\beta}{}^\gamma, \tilde{f}^{\beta\gamma}{}_\alpha)$ together define a Lie bi-algebra. Now applying this to $F_1 = g(x)$, $F_2 = g(y)$ and parametrizing the exchange r-matrix as

$$r^{\alpha\beta} = k^{\kappa\lambda}\zeta^\alpha{}_\kappa\zeta^\beta{}_\lambda \tag{5.11}$$

we find that the infinitesimal action (5.7) will be PL if

$$\zeta^\alpha{}_\kappa\zeta^\beta{}_\lambda\Big(\zeta^\sigma{}_\gamma\mathcal{D}^-_\sigma k^{\kappa\lambda} + k^{\kappa\sigma}f_{\sigma\gamma}{}^\lambda + k^{\sigma\lambda}f_{\sigma\gamma}{}^\kappa + \kappa\tilde{f}^{\kappa\lambda}{}_\gamma\Big) = \frac{1}{2}\Big(\mathcal{D}^{+\alpha}\zeta^\beta{}_\gamma - \mathcal{D}^{+\beta}\zeta^\alpha{}_\gamma\Big)\,. \tag{5.12}$$

It is well-known that all simple PL groups are coboundary. This means that the PL structure on G is given by the Sklyanin bracket,

$$\{h \overset{\otimes}{,} h\} = \frac{1}{\kappa}\left[\hat{R}, h \otimes h\right], \tag{5.13}$$

for which the structure constants of the induced dual Lie algebra are

$$\tilde{f}^{\beta\gamma}{}_{\alpha} = \frac{1}{\kappa}\left(R^{\sigma\beta} f_{\sigma\alpha}{}^{\gamma} + R^{\gamma\sigma} f_{\sigma\alpha}{}^{\beta}\right), \tag{5.14}$$

where $\hat{R} = R^{\alpha\beta} T_\alpha \otimes T_\beta \in \mathcal{G} \wedge G$ is some *constant, antisymmetric* r-matrix. The r-matrix that occurs here is an arbitrary solution of the (modified) classical YB equation,

$$[\hat{R}_{12}, \hat{R}_{23}] + \text{cycl. perm.} = -\nu^2 \hat{f} \tag{5.15}$$

with some constant parameter $-\nu^2$. This parametrization of the right hand side will prove useful below. The value $\nu = 0$ is also allowed and for real Lie groups ν^2 is of course real. From the classification [30, 31] of the solutions we recall that for a compact simple Lie algebra ν must be purely imaginary or zero, while for the maximally non-compact (split) real forms ν is real.

Let us introduce $K^{\alpha\beta}$ by writing

$$k^{\alpha\beta}(M) = K^{\alpha\beta}(M) + R^{\alpha\beta}. \tag{5.16}$$

Then (5.12) can be reduced to

$$\zeta^{\alpha}{}_{\kappa}\zeta^{\beta}{}_{\lambda}\left(\zeta^{\sigma}{}_{\gamma}\mathcal{D}^{-}_{\sigma}K^{\kappa\lambda} + K^{\kappa\sigma}f_{\sigma\gamma}{}^{\lambda} + K^{\sigma\lambda}f_{\sigma\gamma}{}^{\kappa}\right) = \frac{1}{2}\left(\mathcal{D}^{+\alpha}\zeta^{\beta}{}_{\gamma} - \mathcal{D}^{+\beta}\zeta^{\alpha}{}_{\gamma}\right). \tag{5.17}$$

This depends on how the infinitesimal generator X_α is parametrized in terms of $\zeta^{\beta}{}_{\alpha}$, but the explicit reference to the dual structure constants (5.14) has disappeared.

From now on we will concentrate on the simplest case corresponding to the standard action $\zeta^{\beta}{}_{\alpha} = -\delta^{\beta}{}_{\alpha}$. In this case

$$r^{\alpha\beta}(M) = K^{\alpha\beta}(M) + R^{\alpha\beta} \tag{5.18}$$

and (5.17) reduces to

$$\mathcal{D}^{-}_{\alpha}K = [K, \mathcal{T}_\alpha]\,, \qquad \text{where} \quad \mathcal{T}_\alpha := \mathrm{ad}\, T_\alpha\,, \tag{5.19}$$

that is, $(\mathcal{T}_\alpha)_{\sigma}{}^{\lambda} = f_{\sigma\alpha}{}^{\lambda}$. Eq. (5.19) is the infinitesimal form of

$$\hat{K}(hMh^{-1}) = (h \otimes h)\hat{K}(M)(h^{-1} \otimes h^{-1})\,, \tag{5.20}$$

which expresses the equivariance of $\hat{K}(M) = K^{\alpha\beta}(M) T_\alpha \otimes T_\beta$ under the action (5.3). One may also verify directly that the standard left action of G equipped with the PB (5.13) is PL for the exchange r-matrix (5.18) if $\hat{K}(M)$ is equivariant.

So far we have established that the action (5.3) of G on $\check{\mathcal{M}}_{chir}$ is PL if $r(M)$ is the sum of a constant r-matrix R and an equivariant piece $K(M)$. Of course, the exchange r-matrix (5.18) has to be a solution of the dynamical YB equation (3.63). Using (5.19), (3.63) can be rewritten as

$$-(K\mathcal{T}^{\alpha}K)^{\beta\gamma} + \frac{1}{2}\mathcal{D}^{+\alpha}K^{\beta\gamma} + \text{cycl. perm.} = (\nu^2 - \frac{1}{4})f^{\alpha\beta\gamma} \tag{5.21}$$

in this special case, where the cyclic permutations are over the upper indices α, β, γ. The search for solutions of (5.21) is made feasible by the observation that any analytic function of $\mathcal{Y}$ (defined in (3.94)) is equivariant. We show in Appendix B that a solution of (5.21) is given by the analytic function

$$K = \frac{1}{2}\coth\frac{\mathcal{Y}}{2} - \nu\coth(\nu\mathcal{Y}) \ . \tag{5.22}$$

This formula is valid on a domain $\check{G}$ around $e \in G$ where the exponential parametrization is applicable and the power series of the above expression converges. The derivation of the exchange r-matrix given by (5.18), (5.22), which is compatible with the standard action of the PL group G equipped with the PB (5.13), is one of our main results.

We conclude this section with a couple of remarks related to the above r-matrices. We first note that for $\nu = 0$ (5.22) is understood as the appropriate limit and therefore for $R = 0$ we recover from (5.18) the exchange r-matrix r_0 (3.95) together with the classical $\mathcal{G}$-symmetry discussed in sec. 3.5. We can also have $\nu = 0$ in correspondence with any antisymmetric solution $R \neq 0$ of the classical YB equation. For $\nu = 1/2$ we get $K = 0$ and the dynamical r-matrix (5.18) then reduces to one of the constant r-matrices treated in sec. 3.4. For compact groups all solutions (5.22) are strictly dynamical (non-constant), since in this case $-\nu^2 \geq 0$. Finally, we remark that the existence of a suitable local 2-form ρ corresponding to the r-matrix (5.18), (5.22) is guaranteed by the solvability of (3.36).

We have seen that the classical $\mathcal{G}$-symmetries discussed in sec. 3.5 and the special PL symmetries treated in sec. 3.4 are generated by momentum maps. Without going into details, we wish to mention that it is possible to show the existence of a non-Abelian momentum map also in the general case of the above PL symmetries. The momentum map is given by a function $m(M)$ on $\check{\mathcal{M}}_{chir}$ (depending on the monodromy matrix only), which takes its values in the dual PL group G^* in such a way that for all phase space functions F

$$X_\alpha F = \Big(m^{-1}\{m, F\}\Big)_\alpha \,, \tag{5.23}$$

where the $()_\alpha$ component on the right hand side is evaluated in the dual Lie-algebra, $\mathcal{G}^*$. Moreover, the G^*-valued momentum map satisfies the PBs

$$\{m \overset{\otimes}{,} m\} = \eta^*(m) m \otimes m \,, \tag{5.24}$$

where the Poisson tensor $\eta^*(m) \in \mathcal{G}^* \wedge \mathcal{G}^*$ is chosen so that (5.24) defines the Poisson structure of the dual PL group. (For an explanation of these notions, see e.g. [32].)

It is in principle possible to use the the momentum map construction also backwards. If there is a G^*-valued function m on the phase space satisfying (5.24), then using (5.23) to define an infinitesimal generator X_α one obtains that

- the infinitesimal generators represent the Lie algebra, $[X_\alpha, X_\beta] = f_{\alpha\beta}{}^\gamma X_\gamma$,
- the Lie algebra action is PL, i.e., (5.10) holds.

Let now suppose that a *compact* simple Lie algebra $\mathcal{G}$ acts on a phase space as a classical symmetry generated by a $\mathcal{G}^*$-valued, equivariant ('Abelian') momentum map. In this situation one can always define also an infinitesimal PL action of the group G equipped with the so called *standard* PL structure. This is a consequence of the fact [36] that there exists a diffeomorphism between $\mathcal{G}^*$ and G^* that converts the natural linear Poisson structure on $\mathcal{G}^*$ into the standard PL structure on G^*. Applying this to the classical $\mathcal{G}$-symmetry studied in sec. 3.5, we can thus find a map,

$$\mathcal{G}^* \ni \tilde{\Gamma} \mapsto m(\tilde{\Gamma}) \in G^*, \tag{5.25}$$

where $\tilde{\Gamma} : \check{\mathcal{M}}_{chir} \to \mathcal{G}^*$ is given by (3.96), such that $m(\tilde{\Gamma})$ satisfies (5.24) with respect to the standard PL structure. The resulting G^*-valued momentum map then generates a PL action on the phase space $\check{\mathcal{M}}_{chir}$ as outlined above.

This somewhat surprising construction is not specific to the chiral WZNW phase space, since it can be used whenever one has a classical $\mathcal{G}$-symmetry based on a compact simple Lie algebra. When applying it to the chiral WZNW phase space, the Lie algebra action (5.23) constructed using the momentum map (5.25) will be different from the standard one (5.3). It is an interesting question whether this Lie algebra action can be gauge transformed to the standard form and, if the answer is positive, to find the corresponding gauge transform of the r-matrix r_0 (3.95). We wish to discuss this in a future publication.

6 Interpretation in terms of Poisson-Lie groupoids

The CDYB equation (4.32) can be regarded as the guarantee of the Jacobi identity in a PL groupoid [23]. Below we show that eq. (3.63) admits an analogous interpretation. For this, we introduce a family of PL groupoids in such a way that a subfamily of these is naturally associated with the possible PBs on the chiral WZNW phase space. Remarkably, these groupoids are finite dimensional Poisson manifolds that encode practically all information about the infinite dimensional chiral WZNW PBs.

Roughly speaking, a groupoid is a set, say P, endowed with a 'partial multiplication' that behaves similarly to a group multiplication in the cases when it can be performed. To understand the following construction one does not need to know details of the notion of a groupoid (see e.g. [37]), since we shall only use the most trivial example of such a structure, for which

$$P = S \times G \times S = \{(M^F, g, M^I)\}, \tag{6.1}$$

where G is a group and S is some set. The partial multiplication is defined for those triples (M^F, g, M^I) and $(\bar{M}^F, \bar{g}, \bar{M}^I)$ for which $M^I = \bar{M}^F$, and the product is

$$(M^F, g, M^I)(\bar{M}^F, \bar{g}, \bar{M}^I) := (M^F, g\bar{g}, \bar{M}^I) \quad \text{for} \quad M^I = \bar{M}^F. \tag{6.2}$$

In other words, the graph of the partial multiplication is the subset of

$$P \times P \times P = \{(M^F, g, M^I)\} \times \{(\bar{M}^F, \bar{g}, \bar{M}^I)\} \times \{(\hat{M}^F, \hat{g}, \hat{M}^I)\} \tag{6.3}$$

defined by the constraints

$$M^I = \bar{M}^F, \quad \hat{M}^F = M^F, \quad \hat{M}^I = \bar{M}^I, \quad \hat{g} = g\bar{g}, \tag{6.4}$$

where the hatted triple encodes the components of the product. A PL groupoid [38] P is a groupoid and a Poisson manifold in such a way that the graph of the partial multiplication is a *coisotropic* submanifold of $P \times P \times P^-$, where P^- denotes the manifold P endowed with the opposite of the PB on P. In other words, this means that *the constraints that define the graph are first class.* This definition reduces to that of a PL group in the particular case for which the set S in (6.1) consists of a single point.

In the interpretation of (3.70) given in [23] the groupoid P is of the form above with S taken to be a domain in the dual of a Cartan subalgebra of a simple Lie group G. By thinking about a generic monodromy matrix, we now take P to be

$$P = \check{G} \times G \times \check{G}, \tag{6.5}$$

where $\check{G}$ is some open domain in G. On this P, we postulate a PB $\{\ ,\ \}_P$ defined, by using the usual tensorial notation, as follows:

$$\begin{aligned}
&\kappa\{g_1, g_2\}_P = g_1 g_2 \hat{r}(M^I) - \hat{r}(M^F) g_1 g_2 \\
&\kappa\{g_1, M_2^I\}_P = g_1 M_2^I \hat{\Theta}(M^I) \\
&\kappa\{g_1, M_2^F\}_P = M_2^F \hat{\Theta}(M^F) g_1 \\
&\kappa\{M_1^I, M_2^I\}_P = M_1^I M_2^I \hat{\Delta}(M^I) \\
&\kappa\{M_1^F, M_2^F\}_P = -M_1^F M_2^F \hat{\Delta}(M^F) \\
&\kappa\{M_1^I, M_2^F\}_P = 0.
\end{aligned} \tag{6.6}$$

Here κ is an arbitrary constant included for comparison purposes. The 'structure functions' $\hat{r}$, $\hat{\Theta}$, $\hat{\Delta}$ are $\mathcal{G} \otimes \mathcal{G}$ valued functions on $\check{G}$; in components

$$\hat{r}(M) = r^{\alpha\beta}(M) T_\alpha \otimes T_\beta, \quad \hat{\Theta}(M) = \Theta^{\alpha\beta}(M) T_\alpha \otimes T_\beta, \quad \hat{\Delta}(M) = \Delta^{\alpha\beta}(M) T_\alpha \otimes T_\beta. \tag{6.7}$$

It is quite easy to verify that a PB given by the ansatz (6.6) always yields a PL groupoid, since the constraints in (6.4) will be first class for any choice of the structure functions. Of course, the structure functions must satisfy a system of equations in order for the above ansatz to define a PB. The antisymmetry of the PB is ensured by

$$\hat{r} = -\hat{r}_{21} \quad (\hat{r}_{21} := r^{\alpha\beta} T_\beta \otimes T_\alpha) \qquad \text{and} \qquad \hat{\Delta} = -\hat{\Delta}_{21}, \tag{6.8}$$

while the Jacobi identity is, in fact, equivalent to the following system of equations:

$$[\hat{r}_{12}, \hat{r}_{13}] + \Theta_{\alpha\beta} T_1^\alpha \mathcal{R}^\beta \hat{r}_{23} + \text{cycl. perm.} = \mu \hat{f}, \quad \mu = \text{constant}, \tag{6.9}$$

$$[\hat{\Delta}_{12}, \hat{\Delta}_{13}] + \Delta_{\alpha\beta} T_1^\alpha \mathcal{R}^\beta \hat{\Delta}_{23} + \text{cycl. perm.} = 0, \tag{6.10}$$

$$[\hat{r}_{12}, \hat{\Theta}_{13} + \hat{\Theta}_{23}] + [\hat{\Theta}_{13}, \hat{\Theta}_{23}] + \Delta_{\alpha\beta} T_3^\alpha \mathcal{R}^\beta \hat{r}_{12} + \Theta_{\alpha\beta}(T_1^\alpha \mathcal{R}^\beta \hat{\Theta}_{23} - T_2^\alpha \mathcal{R}^\beta \hat{\Theta}_{13}) = 0, \tag{6.11}$$

$$[\hat{\Theta}_{12} + \hat{\Theta}_{13}, \hat{\Delta}_{23}] + [\hat{\Theta}_{12}, \hat{\Theta}_{13}] + \Theta_{\alpha\beta} T_1^\alpha \mathcal{R}^\beta \hat{\Delta}_{23} + \Delta_{\alpha\beta}(T_3^\alpha \mathcal{R}^\beta \hat{\Theta}_{12} - T_2^\alpha \mathcal{R}^\beta \hat{\Theta}_{13}) = 0. \tag{6.12}$$

Observe that the left hand side of (6.9) is of the same form as that of (3.63), but in the groupoid context on the right hand side we have an *arbitrary* constant μ. The derivation of the above equations from the various instances of the Jacobi identity is not difficult. What is somewhat miraculous is that one does not obtain more equations than these. This is actually ensured by our choice of the relationship between the PBs that involve M^I and those that involve M^F. As an illustration, let us explain how (6.9) is derived. By evaluating

$$\{\{g_1, g_2\}_P, g_3\}_P + \text{cycl. perm.} = 0, \tag{6.13}$$

one obtains that this is equivalent to

$$g_1g_2g_3\left([\hat{r}_{12},\hat{r}_{13}]+\Theta_{\alpha\beta}T_1^\alpha\mathcal{R}^\beta\hat{r}_{23}+\text{cycl. perm.}\right)(M^I)=$$
$$=\left([\hat{r}_{12},\hat{r}_{13}]+\Theta_{\alpha\beta}T_1^\alpha\mathcal{R}^\beta\hat{r}_{23}+\text{cycl. perm.}\right)(M^F)g_1g_2g_3. \tag{6.14}$$

This holds if and only if the expression in the parenthesis is a constant, Ad-invariant element of $\wedge^3(\mathcal{G})$, and $\mu\hat{f}$ is the only such element for a simple Lie algebra $\mathcal{G}$.

We have seen that the chiral WZNW PBs are encoded by equations (3.62), (4.16) and (4.17), where $\hat{\Theta}$ and $\hat{\Delta}$ are defined by (3.52) and (3.54) respectively in terms of a solution $\hat{r}$ of (3.63). Now our point is the following: *A PL groupoid can be naturally associated with any Poisson structure on the chiral WZNW phase space by taking the triple $\hat{r}$, $\hat{\Theta}$, $\hat{\Delta}$ that arises in the WZNW model to be the structure functions of a PL groupoid according to (6.6).*

It can be checked that the Jacobi identities of the PL groupoid (6.9)–(6.12) are satisfied for any triple $\hat{r}$, $\hat{\Theta}$, $\hat{\Delta}$ that arises in the WZNW model. This actually follows without any computation since, indeed, the Jacobi identities of the chiral WZNW PBs in (3.62), (4.16), (4.17) lead to the same equations, with $\mu=-\frac{1}{4}$, and they are satisfied since they follow from the symplectic form $\kappa\Omega^\rho_{chir}$.

Among the 'chiral WZNW PL groupoids' described above there are those special cases for which $\hat{K}=\hat{r}-\hat{R}$ satisfies the equivariance condition (5.20) in relation with some constant r-matrix $\hat{R}$ subject to (5.15). In these cases, we equip the group $G=\{h\}$ with the Sklyanin bracket opposite to that in (5.13),

$$\{h\stackrel{\otimes}{,}h\}=\frac{1}{\kappa}\left[h\otimes h,\hat{R}\right], \tag{6.15}$$

and consider its commuting right and left actions on P given respectively by the maps

$$P\times G\ni((M^F,g,M^I),h)\mapsto(M^F,gh,h^{-1}M^Ih)\in P \tag{6.16}$$

and

$$G\times P\ni(h,(M^F,g,M^I))\mapsto(hM^Fh^{-1},hg,M^I)\in P. \tag{6.17}$$

Then it is not difficult to verify that these are *both* Poisson maps, i.e., they define two PL actions of the PL group G (with (6.15)) on the PL groupoid P. In the final analysis, this is a consequence of the fact that, as explained in sec. 5, in the present situation we have a PL action of G on the chiral WZNW phase space whose Poisson structure is encoded by $(P,\{\ ,\ \}_P)$. Here $\check{G}\subset G$ must be Ad-invariant, see footnote 5.

In [23] PL groupoids are associated with arbitrary subalgebras $\mathcal{K}\subset\mathcal{G}$, although the corresponding dynamical r-matrices are described only if $\mathcal{K}$ is a Cartan subalgebra. The $\mathcal{K}=\mathcal{G}$ special case of their groupoids is in fact equivalent to our PL groupoid whose structure function is the r-matrix in (3.95). Their PL groupoids are different from ours in general.

7 Conclusion

In this paper we explored the Poisson structures on the chiral WZNW phase space of group valued quasiperiodic fields with generic monodromy. We have shown that the possible PBs are defined by the exchange r-matrices that are solutions of (3.63). This equation can be viewed as an analogue of the celebrated CDYB equation (3.70), since the latter plays a similar rôle for chiral WZNW fields with diagonal monodromy. An analysis of chiral WZNW Bloch waves and their classical Wakimoto realizations in the spirit of the present paper is contained in [22].

We have given an interpretation of our dynamical YB equation (3.63) in terms of a family of PL groupoids, whose further study may be fruitful. In this respect, the most interesting open questions appear to be to quantize these PL groupoids and to find applications for them outside the chiral WZNW context. It is known that equation (3.70) admits interesting applications in the field of integrable systems [27].

We also investigated the PL symmetries of the exchange algebra (3.62). We have found that for *any* PL structure on the WZNW group G there is a corresponding choice of the exchange r-matrix such that the standard gauge action of G on the chiral fields becomes a PL action. It would be desirable to understand if this result has any analogue at the level of the quantized (chiral) WZNW model.

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A Exchange r-matrices for $SU(2)$

In this appendix we present an explicit, local formula for the most general exchange r-matrix on the simplest compact Lie group $G = SU(2)$. The formula (A.15) below is valid in a neighbourhood of the unit element. It illustrates some general results obtained in sec. 3, and it may prove useful in a future study of the question whether globally defined exchange r-matrices exist for $SU(2)$ or not.

As a basis for the Lie algebra $su(2)$, we choose the generators $T^a := \frac{1}{2i}\sigma_a$, where the σ_a ($a = 1, 2, 3$) are the usual Pauli matrices. We parametrize the matrices $r^{ab}(M)$ and $q^{ab}(M)$ that appear in (3.36) in terms of 3-component vectors as

$$r^{ab} = \epsilon^{abc} R_c \qquad \text{and} \qquad q^{ab} = \epsilon^{abc} Q_c, \tag{A.1}$$

where ϵ^{abc} is the totally antisymmetric tensor for which $\epsilon^{123} = 1$. Furthermore, we identify the $SU(2)$ group manifold with $S^3 \subset \mathbf{R}^4$ by writing $M \in SU(2)$ according to

$$M = x_0\sigma_0 + ix_a\sigma_a\,, \qquad\qquad x_0^2 + x_a x_a = 1, \tag{A.2}$$

whereby x_0, x_a define smooth functions on $SU(2)$ (σ_0 is the 2×2 unit matrix). It is then a matter of straightforward calculation to translate eq. (3.37) into the formula

$$R_a = \frac{1}{4} D^{-1} \left(x_a + 4x_0 Q_a + 4\epsilon^{abc} x_b Q_c \right), \qquad D := x_0 - 4x_a Q_a, \tag{A.3}$$

which is valid on a neighbourhood of the unit element where $D \neq 0$. By assumption, on this neighbourhood the Q_a are smooth functions subject to

$$d\rho(M) = \frac{1}{6}\mathrm{tr}(M^{-1}dM)^{3\wedge} \quad \text{for} \quad \rho(M) = \frac{1}{2}\, q^{ab}(M)\mathrm{tr}(T_a M^{-1}dM) \wedge \mathrm{tr}(T_b M^{-1}dM). \tag{A.4}$$

As discussed in sec. 3.3, (A.4) implies that R_a defined by (A.1), (A.3) yields a solution of the dynamical YB equation (3.63), which for $SU(2)$ can actually be written in the form

$$2R_aR_a + \frac{1}{2}\mathcal{D}^{a+}R_a + \epsilon^{abc}R_c\mathcal{D}_b^- R_a = -\frac{1}{8}. \tag{A.5}$$

Observe that we cannot have a constant solution since the R_a must be real. It is also worth noting that locally we have the inverse of (A.3) given by

$$Q_a = \frac{1}{4}\tilde{D}^{-1}\left(-x_a + 4x_0R_a + 4\epsilon^{abc}x_bR_c\right), \qquad \tilde{D} := x_0 + 4x_aR_a. \tag{A.6}$$

This formula defines via (A.1) a solution of (A.4) out of any solution of (A.5).

We shall now derive the general local solution of (A.5) by making explicit the general local solution (3.93) of (A.4) that we have obtained in sec. 3.5 for any group. For this we need the exponential parametrization of $SU(2)$,

$$M = e^{2\pi\Gamma} = e^{2\pi\Gamma_aT^a} \qquad \text{with} \qquad |\Gamma|^2 = \Gamma_a\Gamma_a < 1, \tag{A.7}$$

which covers the domain $SU(2)\setminus\{-\sigma_0\}$. Upon comparison with (A.2), we get

$$x_0 = \cos(\pi|\Gamma|), \qquad x_a = -\frac{\Gamma_a}{|\Gamma|}\sin(\pi|\Gamma|). \tag{A.8}$$

We shall also use the following expressions for the powers of the operator $\mathrm{ad}\Gamma$. For the odd powers, we have

$$(\mathrm{ad}\Gamma)^{2n+1} = (-1)^n|\Gamma|^{2n}(\mathrm{ad}\Gamma), \quad n\geq 0, \qquad (\mathrm{ad}\Gamma)(T^a) = [\Gamma, T^a] = \epsilon^{abc}T_b\Gamma_c. \tag{A.9}$$

For the even powers, defining the matrix of $(\mathrm{ad}\Gamma)^n$ by $(\mathrm{ad}\Gamma)^n(T^b) = [(\mathrm{ad}\Gamma)^n]_a{}^bT^a$, we have

$$[(\mathrm{ad}\Gamma)^{2n}]_a{}^b = (-1)^n|\Gamma|^{2n}\left(\delta_{ab} - \frac{\Gamma_a\Gamma_b}{|\Gamma|^2}\right), \quad n\geq 1. \tag{A.10}$$

Using these relations, we can rewrite the formulae (3.94) and (3.95) as follows:

$$q_0^{ab} = \epsilon^{abc}Q_c^{(0)} \quad \text{with} \quad Q_c^{(0)} = \Gamma_c\frac{2\pi|\Gamma| - \sin(2\pi|\Gamma|)}{8|\Gamma|\sin^2(\pi|\Gamma|)}\,, \tag{A.11}$$

and

$$r_0^{ab} = \epsilon^{abc}R_c^{(0)} \quad \text{with} \quad R_c^{(0)} = \frac{1}{4}\frac{\Gamma_c}{|\Gamma|}\Big(\cot(\pi|\Gamma|) - \frac{1}{\pi|\Gamma|}\Big)\,. \tag{A.12}$$

One may check that eqs. (A.3)–(A.6) hold for these expressions, which represent smooth functions on $SU(2)\setminus\{-\sigma_0\}$.

To obtain the most general 2-form ρ on $SU(2)\setminus\{-\sigma_0\}$ that satisfies (A.4), we have to add an arbitrary closed 2-form to the 2-form, ρ_0, that corresponds to the matrix q_0^{ab}. In fact, the result can be written as

$$\rho(\Gamma) = d\Gamma_a\wedge d\Gamma_b\epsilon^{cba}(\Gamma_c\frac{\sin(2\pi|\Gamma|) - 2\pi|\Gamma|}{2|\Gamma|^3} + U_c(\Gamma))\,, \tag{A.13}$$

where $U_a(\Gamma)$ is a smooth 'vector-function' in the interior of the unit ball, $|\Gamma| < 1$, which is divergence free, i.e.,

$$\sum_{a=1}^{3} \frac{\partial U_a(\Gamma)}{\partial \Gamma_a} = 0. \tag{A.14}$$

We then have to rewrite this 2-form in the manner indicated by the second parts of (A.1) and (A.4). By means of (A.3), this will provide us with the general local solution of (A.5). By performing the necessary (rather tedious) calculations, in this way we obtain the following formula:

$$R_a = \frac{\Gamma_a}{|\Gamma|}\left[\frac{1}{4}\cot(\pi|\Gamma|) - \frac{1}{|\Gamma|(4\pi - 2\Gamma\cdot U)}\right] + \left[\frac{\Gamma_a}{|\Gamma|}(\Gamma\cdot U) - |\Gamma| U_a\right]\frac{1}{(4\pi - 2\Gamma\cdot U)2\pi|\Gamma|}, \tag{A.15}$$

where $\Gamma \cdot U = \Gamma_a U_a$. This expression is valid on the open subset of $SU(2)$ that excludes $-\sigma_0$ and the points where $\Gamma \cdot U = 2\pi$. In particular, R_a is smooth in a neighbourhood of the unit element, for which $\Gamma = 0$, since $U_a(\Gamma)$ is smooth there by assumption. One can also verify explicitly that on its domain of validity R_a solves the dynamical YB equation (A.5) for any divergence free $U_a(\Gamma)$. For this verification, one needs to spell out (A.5) more explicitly. For instance, if one uses the x_a in (A.2) as coordinates around $\sigma_0 \in SU(2)$, then (A.5) becomes

$$2R_aR_a - \frac{\sqrt{1 - x_b x_b}}{2}\frac{\partial R_a}{\partial x_a} + 2x_a R_b \frac{\partial R_a}{\partial x_b} - 2R_b x_b \frac{\partial R_a}{\partial x_a} = -\frac{1}{8}. \tag{A.16}$$

In summary, we have derived the form of the most general exchange r-matrix in a neighbourhood of the unit element of $SU(2)$. The solution (A.15) may in general develop singularities away from the unit element, and it is an open question if globally smooth solutions of (A.5) exist on $SU(2)$ or not.

B Analytic solution of the dynamical YB equation

In this appendix we show that the equivariant analytic function (5.22) is a solution of the dynamical YB equation (5.21).

We will use the power series expansion of the coth function:

$$\coth z = \frac{1}{z} + \sum_{r=0}^{\infty} \alpha_r z^{2r+1}\,. \tag{B.1}$$

Here the coefficients α_r can be expressed in terms of the Bernoulli numbers [35]. They can also be computed using the recursion relation

$$\alpha_0 = \frac{1}{3}\,, \qquad \alpha_m = -\frac{1}{2m+3}\sum_{r=0}^{m-1}\alpha_r\alpha_{m-1-r}\,, \qquad m = 1, 2, \ldots \tag{B.2}$$

Using the properties of the operator $\mathcal{D}^{+\alpha}$ (3.7) and the definition of K (5.22) the first two terms of (5.21) can be expanded as

$$(K\mathcal{T}^\alpha K)^{\beta\gamma} = \sum_{r,s=0}^{\infty} \alpha_r\alpha_s\Big(\frac{1}{2^{2r+2}} - \nu^{2r+2}\Big)\Big(\frac{1}{2^{2s+2}} - \nu^{2s+2}\Big)\Big(\mathcal{Y}^{2s+1}\mathcal{T}^\alpha\mathcal{Y}^{2r+1}\Big)^{\beta\gamma} \tag{B.3}$$

and

$$\frac{1}{2}\mathcal{D}^{+\alpha}K^{\beta\gamma} = \sum_{r=0}^{\infty}\alpha_r\Big(\frac{1}{2^{2r+2}} - \nu^{2r+2}\Big)\sum_{s=0}^{2r}\Big(\frac{\mathcal{Y}}{2}\coth\frac{\mathcal{Y}}{2}\Big)^{\alpha\lambda}\Big(\mathcal{Y}^s\mathcal{T}_\lambda\mathcal{Y}^{2r-s}\Big)^{\beta\gamma}. \tag{B.4}$$

Here, in writing the second equality we exploited that in the parametrization of M introduced in sec. 3.5 one readily obtains by writing $\mathcal{Y} = \mathcal{X}^a\mathcal{T}_a$ that

$$\mathcal{L}^\alpha(\mathcal{X}^\beta) = \Big(\frac{\mathcal{Y}}{1-\exp(-\mathcal{Y})}\exp(-\mathcal{Y})\Big)^{\alpha\beta}, \qquad \mathcal{R}^\alpha(\mathcal{X}^\beta) = \Big(\frac{\mathcal{Y}}{\exp(\mathcal{Y})-1}\exp(\mathcal{Y})\Big)^{\alpha\beta}. \tag{B.5}$$

It is clear that all terms in (5.21) are built from powers of $\mathcal{Y}$ and structure constants. It will prove useful to contract all the indices with Lie algebra generators and thus reformulate (5.21) as an equation in the triple tensor product of the Lie algebra. We introduce the notation

$$\langle k,l,m\rangle := (\mathcal{Y}^k)^{\alpha\kappa}(\mathcal{Y}^l)^{\beta\lambda}(\mathcal{Y}^m)^{\gamma\sigma}f_{\kappa\lambda\sigma}\,T_\alpha\otimes T_\beta\otimes T_\gamma\,. \tag{B.6}$$

Cyclic permutation of the indices now corresponds to cyclic permutation of the tensor factors and we also introduce the symbol

$$[k,l,m] := \langle k,l,m\rangle + \langle l,m,k\rangle + \langle m,k,l\rangle\,. \tag{B.7}$$

We expand both sides of (5.21) in powers of ν and every coefficient of ν also in powers of $\mathcal{Y}$. In our notation this latter expansion corresponds to putting together all terms with a fixed total $N = k+l+m$, and eq. (5.21) requires separately the equality of all such terms on the two sides.

We start with the ν^0 terms. We find that the $N=0$ piece is satisfied identically, while for $N = 2m+2$ ($m = 0,1,\ldots$) we get

$$\begin{aligned}\alpha_{m+1}\sum_{s=0}^{2m+2}(-1)^{s+1}[0,s,2m+2-s] - \sum_{r+s=m}\alpha_r\alpha_s[0,2s+1,2r+1]\\ + \sum_{r+k=m}\alpha_r\alpha_k\sum_{s=0}^{2r}(-1)^{s+1}[2k+2,s,2r-s] = 0\,.\end{aligned} \tag{B.8}$$

Similarly the $N=0$ piece of the ν^2 term vanishes identically, while for $N = 2r+2$ ($r = 0,1,\ldots$) we obtain

$$[2r+2,0,0] + [0,1,2r+1] + [0,2r+1,1] = 0\,. \tag{B.9}$$

Finally, from the ν^{2m+4} terms (for $m = 0,1,\ldots$) we get contributions with $N = (2m+2)$-th powers of $\mathcal{Y}$,

$$\alpha_{m+1}\sum_{s=0}^{2m+2}(-1)^s[0,s,2m+2-s] - \sum_{r+s=m}\alpha_r\alpha_s[0,2s+1,2r+1] = 0, \tag{B.10}$$

and also terms with $N = (2m+4+2r)$-th powers of $\mathcal{Y}$ (for $r = 0,1,\ldots$),

$$[0,2m+3,2r+1] + [0,2r+1,2m+3] + \sum_{s=0}^{2m+2}(-1)^s[2r+2,s,2m+2-s] = 0\,. \tag{B.11}$$

Before proceeding we note that using the Jacobi identity for the structure constants we can write down the following identity:

$$\langle k+1,l,m\rangle + \langle k,l+1,m\rangle + \langle k,l,m+1\rangle = 0\,. \tag{B.12}$$

Now it is easy to see that (B.9) is a special case and (B.11) is a simple consequence of the above identity. In fact, to prove (B.11) we group $[0,2m{+}3,2r{+}1]$ with the first ($[0,2r{+}1,2m{+}3]$ with the last) $m+1$ terms of the sum:

$$\begin{aligned} Z_1 &= [0,2m+3,2r+1] + \sum_{s=0}^{m}(-1)^s[2r+2,s,2m+2-s],\\ Z_2 &= [0,2r+1,2m+3] + \sum_{j=0}^{m}(-1)^j[2r+2,2m+2-j,j]\,, \end{aligned}$$

then in both groups, for odd s (j) we use (B.12) to write

$$\begin{aligned} (-1)^s[2r+2,s,2m+2-s] &= [2r+1,s+1,2m+2-s] + [2r+1,s,2m+3-s],\\ (-1)^j[2r+2,2m+2-j,j] &= [2r+1,2m+3-j,j] + [2r+1,2m+2-j,j+1]. \end{aligned}$$

In this form, as a consequence of (B.12), both Z_1 and Z_2 cancel 'telescopically' almost completely; for odd m they give

$$Z_1 \mapsto [2r+1,m+1,m+2], \qquad Z_2 \mapsto [2r+1,m+2,m+1], \tag{B.13}$$

while for even m we get

$$\begin{aligned} Z_1 &\mapsto [2r+2,m,m+2] + [2r+1,m,m+3],\\ Z_2 &\mapsto [2r+2,m+2,m] + [2r+1,m+3,m]\,. \end{aligned} \tag{B.14}$$

One readily verifies, using again (B.12), that these remaining terms give zero in both cases with the 'central' element of the sum $(-1)^{m+1}[2r+2,m+1,m+1]$.

As for (B.8), it should be perfectly possible to prove it using the properties of the coefficients α_r, but in the present context there is no need to prove it independently since we know that the $\nu = 0$ case coincides with the solution (3.95) and therefore it must satisfy (3.63).

Thus we are left with (B.10). Using (B.8), we can rewrite it as

$$2\sum_{r+k=m}\alpha_r\alpha_k[0,2r+1,2k+1] + \sum_{r+k=m}\alpha_r\alpha_k\sum_{s=0}^{2r}(-1)^s[2k+2,s,2r-s] = 0 \tag{B.15}$$

and in this form we see that it trivially follows from (B.11) and the index symmetry of $\alpha_k\alpha_r$.

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